Fun with Binomials

Binomials are numbers that give the coefficients of the terms of the expansion of $(a+b)^n$. Binomials have many interest properties. Two of those properties will be looked at as well as a curiosity relating to Fermat's Last Theorem.

Fermat's Last Theorem states that for $n > 2$ there are no integer solutions to the equation $c^n = a^n + b^n$. Fermat wrote in a book he was reading that he had a proof of the conjecture but that the margin in the book was not wide enough to write the proof in. No one knows what proof he had in mind. Over several centuries many proofs have been put forward, but all were found to have flaws. In 1994 the conjecture was proven, but it was given by one who scales the mountain peaks of the math world and is only understood by those who hike the high places above the clouds. This would not have been what Fermat envisioned as a proof. No one would have been walking the high peaks necessary for such a proof in his time. Is there a proof that he would have been capable of giving? It is believed not and no simple proof is given here. If there is any simple proof it would likely be related to binomials. Two relations relating to binomials will be looked at.

Relation 1

The first relation is

$$(R1) \qquad \sum_{i=0}^{\frac{n-1}{2}} (-1)^i (n-2i) \binom{n}{i} = 0 \quad where\ n\ is\ odd.$$

The term $\binom{n}{i}$ is the $(i^{th}+1)$ value of the binomial expansion and is given by $\frac{n!}{(n-i)!i!}$. These are the coefficients of the expansion of $(a+b)^n$ with the first coefficient being $\binom{n}{0}$. What is Relation R1 saying? The notation while being concise and elegant somewhat hides what is physically, in a mathematical sense, being stated. Go back to the 8th or 9th grade, at least in my time, and construct a binomial cascade up to $n = 7$.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Relation (R1) states that for $n = 7$

$$(1)(7)(1) + (-1)(5)(7) + (1)(3)(21) + (-1)(1)(35) = 0.$$

Doing the arithmetic one finds that $7 - 35 + 63 - 35$ does equal zero. To prove (R1) in general the method of the construction of the binomial cascade will be put in use.

The first binomial value of the n^{th} row is $\binom{n}{0} = 1$. The last value of the row is also 1. The values between the first and the last are found by adding the values in the preceding row just above and to either side of the value being looked for, i.e,

$$\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i} .$$

This construction can be done again to obtain the n^{th} row in terms of the $(n-2)$ row. The result is that the first term of the n^{th} row is $\binom{n}{0} = \binom{n-2}{0} = 1$. The second term is $\binom{n}{1} = 2\binom{n-2}{0} + \binom{n-2}{1}$. The middle terms are $\binom{n}{i} = \binom{n-2}{i-2} + 2\binom{n-2}{i-1} + \binom{n-2}{i}$. Applying the above to (R1) gives, remembering that n is odd

$$(a)\quad \sum_{i=0}^{\frac{n-1}{2}}(-1)^i(n-2i)\binom{n}{i} = n\binom{n-2}{0} + (n-2)\left[2\binom{n-2}{0}+\binom{n-2}{1}\right]$$

$$+\sum_{i=2}^{\frac{n-1}{2}}(-1)^i(n-2i)\left[\binom{n-2}{i-2}+2\binom{n-2}{i-1}+\binom{n-2}{i}\right]$$

$$(b)\quad \sum_{i=0}^{\frac{n-1}{2}}(-1)^i(n-2i)\binom{n}{i} = n\binom{n-2}{0} + (n-2)\left[2\binom{n-2}{0}+\binom{n-2}{1}\right]$$

$$+\sum_{i=2}^{\frac{n-1}{2}}(-1)^i(n-2i)\left[\binom{n-2}{i-2}\right]+2\sum_{i=2}^{\frac{n-1}{2}}(-1)^i(n-2i)\left[\binom{n-2}{i-1}\right]+\sum_{i=2}^{\frac{n-1}{2}}(-1)^i(n-2i)\left[\binom{n-2}{i}\right]$$

The letter "i" is index variable for the summation. It can be changed as long as it is likewise changed over the summation. In the first summation leave i as it is. In the second summation let $i = j - 1$. In the third summation let $i = k - 2$.

$$\sum_{i=0}^{\frac{n-1}{2}}(-1)^i(n-2i)\binom{n}{i} = n\binom{n-2}{0} + (n-2)\left[2\binom{n-2}{0}+\binom{n-2}{1}\right]$$

$$+\sum_{i=2}^{\frac{n-1}{2}}(-1)^i(n-2i)\binom{n-2}{i-2}+2\sum_{j=3}^{\frac{n+1}{2}}(-1)^{j-1}(n-2j+2)\binom{n-2}{j-2}+\sum_{k=4}^{\frac{n+3}{2}}(-1)^k(n-2k+4)\binom{n-2}{k-4}$$

Let the three summations be s1, s2 and s3, respectively as given in the above equation and rewrite them as below.

$$s1 = \sum_{i=2}^{\frac{n-1}{2}} (-1)^i (n-2i) \binom{n-2}{i-2} = (n-4)\binom{n-2}{0} - (n-6)\binom{n-2}{1} + \sum_{i=4}^{\frac{n-1}{2}} (-1)^i (n-2i) \binom{n-2}{i-2}$$

$$s2 = 2\sum_{j=3}^{\frac{n+1}{2}} (-1)^{j-1} (n-2j+2) \binom{n-2}{j-2} = 2(n-4)\binom{n-2}{1} + 2\sum_{j=4}^{\frac{n-1}{2}} (-1)^{j-1} (n-2j+2) \binom{n-2}{j-2} + 2(-1)^{\frac{n-1}{2}} \binom{n-2}{\frac{n-3}{2}}$$

$$s3 = \sum_{k=4}^{\frac{n+3}{2}} (-1)^k (n-2k+4) \binom{n-2}{k-4} = \sum_{k=4}^{\frac{n-1}{2}} (-1)^k (n-2k+4) \binom{n-2}{k-2} + (-1)^{\frac{n+1}{2}}(3)\binom{n-2}{\frac{n-3}{2}} + (-1)^{\frac{n+3}{2}}(1)\binom{n-2}{\frac{n-1}{2}}$$

Let the summation terms in each of s1, s2 and s3 be s11, s22 and s33, respectively. Change index j to i and index k to i. These are just indices and be any letter. Combining the summations gives

$$\begin{aligned} s11 + s22 + s33 &= \sum_{i=4}^{\frac{n-1}{2}} [(-1)^i (n-2i) + 2(-1)^{i-1}(n-2i+2) + (-1)^i (n-2i+4)] \binom{n-2}{i-2} \\ &= (-1)^i \sum_{i=4}^{\frac{n-1}{2}} [(n-2i) - 2(n-2i+2) + (n-2i+4)] \binom{n-2}{i-2} \\ &= (-1)^i \sum_{i=4}^{\frac{n-1}{2}} [(n-2n+n) - (-2i+4i-2i) + (-4+4)] \binom{n-2}{i-2} \\ &= (-1)^i \sum_{i=4}^{\frac{n-1}{2}} [0] \binom{n-2}{i-2} \\ &= 0 \end{aligned}$$

Equation (b) can be written

$$\begin{aligned} \sum_{i=0}^{\frac{n-1}{2}} (-1)^i (n-2i) \binom{n}{i} &= n\binom{n-2}{0} + (n-2)\left[2\binom{n-2}{0} + \binom{n-2}{1}\right] + s1 + s2 + s3 \\ &= n\binom{n-2}{0} + (n-2)\left[2\binom{n-2}{0} + \binom{n-2}{1}\right] + (n-4)\binom{n-2}{0} - (n-6)\binom{n-2}{1} + 2(n-4)\binom{n-2}{1} \\ &\quad + 2(-1)^{\frac{n-1}{2}} \binom{n-2}{\frac{n-3}{2}} + (-1)^{\frac{n+1}{2}}(3)\binom{n-2}{\frac{n-3}{2}} + (-1)^{\frac{n+3}{2}}(1)\binom{n-2}{\frac{n-1}{2}} \end{aligned}$$

Combining binomial terms gives

$$\sum_{i=0}^{\frac{n-1}{2}}(-1)^i(n-2i)\binom{n}{i}$$

$$= [(n-2(n-2)+(n-4)]\binom{n-2}{0} + [-(n-2)-(n-6)+2(n-4)]\binom{n-2}{1} + (-1)^{\frac{n+3}{2}}\binom{n-2}{\frac{n-1}{2}} + \left[2(-1)^{\frac{n-1}{2}} + 3(-1)^{\frac{n+1}{2}}\right]\binom{n-2}{\frac{n-3}{2}}$$

$$= [0]\binom{n-2}{0} + [0]\binom{n-2}{1} + (-1)^{\frac{n-1}{2}}\left[2(-1)^2\binom{n-2}{\frac{n-1}{2}} + (1+3(-1)^1)\binom{n-2}{\frac{n-3}{2}}\right]$$

$$= 2(-1)^{\frac{n-1}{2}}\left[\binom{n-2}{\frac{n-1}{2}} - \binom{n-2}{\frac{n-3}{2}}\right]$$

$$= 2(-1)^{\frac{n-1}{2}}\left[\frac{(n-2)!}{\left(n-2-\frac{n-1}{2}\right)!\left(\frac{n-1}{2}\right)!} - \frac{(n-2)!}{\left(n-2-\frac{n-3}{2}\right)!\left(\frac{n-3}{2}\right)!}\right]$$

$$= 2(-1)^{\frac{n-1}{2}}\left[\frac{(n-2)!}{\left(\frac{n-3}{2}\right)!\left(\frac{n-1}{2}\right)!} - \frac{(n-2)!}{\left(\frac{n-1}{2}\right)!\left(\frac{n-3}{2}\right)!}\right]$$

$$= 0$$

This proves relation (R1).

The trek taken has not been a bush whack but has been a swamp walk. For those who prefer less rigor and more clarity take the terms from equation (a) and put them in a table the columns of which are the binomial coefficients. The table will look as below.

$\binom{n-2}{0}$	$\binom{n-2}{1}$	$\binom{n-2}{2}$	$\cdots$	$\binom{n-2}{k}$	$\cdots$	$\binom{n-2}{\frac{n-3}{2}}$	$\binom{n-2}{\frac{n-1}{3}}$
n	$-(n-2)$	$n-4$	$\cdots$	$(-1)^k(n-2k)$	$\cdots$		
$-2(n-2)$	$2(n-4)$	$-2(n-6)$	$\cdots$	$2(-1)^{k+1}(n-2k+2)$	$\cdots$	$(-1)^{\frac{n-3}{2}}(3)$	
$n-4$	$-(n-6)$	$n-8$	$\cdots$	$(-1)^{k+2}(n-2k-4)$	$\cdots$	$(-1)^{\frac{n-1}{2}}(2)$	$(-1)^{\frac{n-1}{2}}$
0	0	0	$\cdots$	0	$\cdots$		

Summing the columns up to the last two gives the zeros found in the more rigorous proof. The last two columns are handled as in the more rigorous proof.

This looked messy but was not that difficult.

Relation 2

The second relation is

$$(R2) \qquad (a^n + b^n) = -\sum_{i=0}^{\frac{n-1}{2}} \beta_i(1)(ab)^i(a+b)^{n-2i}$$

The β_i terms will be explained in the construction of the proof. The proof will be done by construction which is permissible for those who hike the lower hills of the math world. To make the construction less messy define

$$\beta_1(k) = \binom{n}{k} \text{ and } A(x,y) = a^x b^y + a^y b^x \ .$$

Two relations involving A that will be helpful are

$(a1) \quad A(x,0) = a^x + b^x$ and

$(a2) \quad A(x,y) = (ab)A(x-1,y-1)$.

The trek will not be up a mountain, but it will be over some uneven ground with maybe a little bushwhacking. After a struggle over the terrain, a look back will give a clearer view at what was transverse. Start with

$$(a+b)^n = \sum_{i=0}^{\frac{n-1}{2}} \beta_1(i)(ab)^i(a+b)^{n-2i} \qquad where \ \ \beta_1(k) = \binom{n}{k}$$

This is the formula for the binomial expansion taking advantage of its symmetry. The $\beta_1(k)$ terms are the binomial coefficients. Continuing

$$a^n + b^n = a^n + b^n + \sum_{k=1}^{\frac{n-1}{2}} [\beta_1(k)A(n-k,k) - \beta_1(k)A(n-k,k)]$$

$$= (a+b)^n - \sum_{k=1}^{\frac{n-1}{2}} \beta_1(k)A(n-k,k)$$

$$= (a+b)^n - \sum_{j=0}^{\frac{n-3}{2}} \beta_1(j+1)A(n-j-1,j+1) \qquad where\ \ j = k-1$$

$$= (a+b)^n - (ab)\sum_{j=0}^{\frac{n-3}{2}} \beta_1(j+1)A(n-2-j,j)$$

$$= (a+b)^n - (ab)\left[\beta_1(1)(a^{n-2}+b^{n-2}) + \sum_{j=0}^{\frac{n-3}{2}} \left\{\beta_1(1)\binom{n-2}{j} + \left[\beta_1(j+1) - \beta_1(1)\binom{n-2}{j}\right]\right\} A(n-2-j,j)\right]$$

$$= (a+b)^n - \beta_1(1)(ab)(a+b)^{n-2} - (ab)\sum_{j=0}^{\frac{n-3}{2}} \beta_2(j)A(n-2-j,j) \quad where \quad \beta_2(j) = \beta_1(j+1) - \beta_1(1)\binom{n-2}{j} \tag{1}$$

$$= -\sum_{i=0}^{1} \beta_i(1)(ab)^i(a+b)^{n-2i} - (ab)\sum_{k=1}^{\frac{n-3}{2}} \beta_2(k+1)A(n-3-k,k+1) \quad where \quad \beta_0(1) = -1 \quad and \quad k = j-1$$

$$= -\sum_{i=0}^{1} \beta_i(1)(ab)^i(a+b)^{n-2i} - (ab)^2\sum_{k=1}^{\frac{n-5}{2}} \beta_2(k+1)A(n-4-k,k)$$

$$= -\sum_{i=0}^{1} \beta_i(1)(ab)^i(a+b)^{n-2i}$$

$$- (ab)^2\left[\beta_2(1)(a^{n-4}+b^{n-4}) + \sum_{k=0}^{\frac{n-5}{2}} \left\{\beta_2(1)\binom{n-4}{k} + \left[\beta_2(k+1) - \beta_2(1)\binom{n-4}{k}\right]\right\} A(n-4-k,k)\right]$$

$$= -\sum_{i=0}^{2} \beta_i(1)(ab)^i(a+b)^{n-2i}$$

$$- (ab)^2\sum_{k=0}^{\frac{n-5}{2}} \beta_3(j)A(n-2-k,k) \quad where \quad \beta_3(k) = \beta_2(k+1) - \beta_2(1)\binom{n-4}{k} \tag{2}$$

$$\vdots$$

$$= -\sum_{i=0}^{m} \beta_i(1)(ab)^i(a+b)^{n-2i} - (ab)^m \sum_{k=0}^{\frac{n-2m-1}{2}} \beta_{m+1}(j)A(n-m-k,k) \text{ , } \beta_{m+1}(k) = \beta_m(k+1) - \beta_m(1)\binom{n-2m}{k}, m < \frac{n-3}{2} \qquad (m)$$

$$\vdots$$

$$= -\sum_{i=0}^{\frac{n-1}{2}} \beta_i(1)(ab)^i(a+b)^{n-2i} - (ab)^{\frac{n-1}{2}} \sum_{k=0}^{0} \beta_{\frac{n+1}{2}}(k)A\left(\frac{n+1}{2}-k,k\right) \text{ , } \beta_{\frac{n+1}{2}}(k) = \beta_{\frac{n-1}{2}}(k+1) - \beta_{\frac{n-1}{2}}(1)\binom{1}{k} \qquad \left(\frac{n-1}{2}\right)$$

$$\left[\beta_{\frac{n+1}{2}}(k)\right]_{k=0} = \left[\beta_{\frac{n-1}{2}}(k+1) - \beta_{\frac{n-1}{2}}(1)\binom{1}{k}\right]_{k=0} = \beta_{\frac{n-1}{2}}(1) - \beta_{\frac{n-1}{2}}(1)\binom{1}{0} = 0$$

$$Thus \quad (a^n+b^n) = -\sum_{i=0}^{\frac{n-1}{2}} \beta_i(1)(ab)^i(a+b)^{n-2i} \quad where \ \beta_0(1) = -1, \quad \beta_{m+1}(k) = \beta_m(k+1) - \beta_m(1)\binom{n-2m}{k}$$

Below is a Matlab program to compute the $\beta_i(1)$ values for a given n, $n = 9$ in the program.

```
n = 9;
b = zeros((n-1)/2,(n+1)/2);
for k = 1:5
    b(1,k) = -bi(n,k);
end
for j = 2:((n+1)/2)
    for k = 1:((n+1)/2-(j-1))
        b(j,k) = b(j-1,k+1) - b(j-1,1)*bi(n-2*(j-1),k);
    end
end
b = [[1 zeros(1,(n-1)/2)]; b];
a = fliplr(rot90(b,3));
a2 = [fliplr(1:2:n) 0];
a = [a2;a(1,:)]
```

Now, to look back to get a clearer view of the terrain, take $n = 9$ and construct the β values without using any symbolism to condense the terms such as summation sings. Start with

$$a^9 + b^9 = a^9 + b^9$$

Add and subtract the middle values for the binomial expansion of $(a + b)^9$.

$$a^9 + b^9 = (a^9 + b^9) + [9(a^8b + ab^8) + 36(a^7b^2 + a^2b^7) + 84(a^6b^3 + a^3b^6) + 126(a^5b^4 + a^4b^5)] \\ - [9(a^8b + ab^8) + 36(a^7b^2 + a^2b^7) + 84(a^6b^3 + a^3b^6) + 126(a^5b^4 + a^4b^5)]$$

Combine the first two terms with those in the first brackets and divide out ab from the terms in the second brackets.

$$= (a + b)^9 - (ab)[9(a^7 + b^7) + 36(a^6b + ab^6) + 84(a^5b^2 + a^2b^5) + 126(a^4b^3 + a^3b^4)]$$

Within the brackets add 9 times the middle terms in the binomial expansion of $(a + b)^7$ and subtract those terms from the like terms.

$$= (a + b)^9 - (ab)[9(a^7 + b^7) + 63(a^6b + ab^6) + 189(a^5b^2 + a^2b^5) + 315(a^4b^3 + a^3b^4)] \\ - (ab)[(36 - 63)(a^6b + ab^6) + (84 - 189)(a^5b^2 + a^2b^5) + (126 - 315)(a^4b^3 + a^3b^4)]$$

$$= (a + b)^9 - 9(ab)(a + b)^7 + (ab)[27(a^6b + ab^6) + 105(a^5b^2 + a^2b^5) + 189(a^4b^3 + a^3b^4)]$$

$$= (a + b)^9 - 9(ab)(a + b)^7 + (ab)^2[27(a^5 + b^5) + 105(a^4b + ab^4) + 189(a^3b^2 + a^2b^3)]$$

$$= (a + b)^9 - 9(ab)(a + b)^7 + (ab)^2[27(a^5 + b^5) + 135(a^4b + ab^4) + 270(a^3b^2 + a^2b^3)] \\ + (ab)^2[(105 - 135)(a^4b + ab^4) + (189 - 270)(a^3b^2 + a^2b^3)]$$

$$= (a + b)^9 - 9(ab)(a + b)^7 + 27(ab)^2(a + b)^5 + (ab)^2[-30(a^4b + ab^4) - 81(a^3b^2 + a^2b^3)]$$

$$= (a + b)^9 - 9(ab)(a + b)^7 + 27(ab)^2(a + b)^5 - (ab)^3[30(a^3 + b^3) - 81(a^2b + ab^2)]$$

$$= (a + b)^9 - 9(ab)(a + b)^7 + 27(ab)^2(a + b)^5 - (ab)^3[30(a^3 + b^3) + 90(a^2b + ab^2)] \\ - (ab)^3[(81 - 90)(a^2b + ab^2)]$$

$$= (a + b)^9 - 9(ab)(a + b)^7 + 27(ab)^2(a + b)^5 - 30(ab)^3(a + b)^3 + 9(ab)^4(a + b)$$

From the Matlab program above with $n = 9$ one gets the same β values as were just derived.

Look at the relation $c^n = a^n + b^n$ where n is an odd positive integer greater than 1 In Relation 2, $\beta_{\frac{n-1}{2}} = n$. This is left to the reader to show. The relation can be reduced to

$$a^n + b^n = \delta \times (a+b)^2 + n \times (a+b) \times (ab)^{\frac{n-1}{2}}$$

for some integer δ . The term $(a+b)$ can be factored out giving

$$a^n + b^n = (a+b)\left(\delta \times (a+b) + n \times (ab)^{\frac{n-1}{2}}\right).$$

Fermat's Last Theorem centers on the equation

$c^n = a^n + b^n$ where c, b, a, n are positive integers.

The theorem states that no positive integers c, b, a exist for $n > 2$ such that the equation holds. No generality is lost confining n to being a positive odd prime number and factoring out any common factors from c, b, a so that they have no factors in common.

One may write

$$c^n = \; a^n + b^n = (a+b)\left(\delta \times (a+b) + n \times (ab)^{\frac{n-1}{2}}\right).$$

Assume that n is not a factor of c, i.e. $n \nmid c$. Also assume that p is a factor of c, i.e. $p|c$. Since p is a factor of c , it is not a factor of $a \; or \; b$. If p is a factor of $(a+b)$, then it cannot be a factor of $\left(\delta \times (a+b) + n \times (ab)^{\frac{n-1}{2}}\right)$ since it is a factor of $n \times (ab)^{\frac{n-1}{2}}$. All instances of p must be in $(a+b)$. That is p^n is in $(a+b)$. This is true for all factors of $(a+b)$. Thus

$$a + b = \gamma^n \; for \; some \; integer \; \gamma.$$

Since n is an odd integer in Relation 2, replace a with c and b with $-a \; or -b$. Using the same construction and arguments

$$c - a = \beta^n \; for \; some \; integer \; \beta$$

$$c - b = \alpha^n \; for \; some \; integer \; \alpha.$$

Solving $a, b\ and\ c$ gives

$$a = \frac{\gamma^n - (\beta^n - \alpha^n)}{2}$$

$$b = \frac{\gamma^n + (\beta^n - \alpha^n)}{2}$$

$$c = \frac{\gamma^n + (\beta^n + \alpha^n)}{2}$$

where n is an odd integer greater than 1 and all other terms are positive integers. To sum up, if $a, b\ and\ c$ are positive integers such that $c^n = b^n + a^n$ where n is a positive odd integer greater than $1,$ then there exists integers $\alpha, \beta\ and\ \gamma$ such that the above relations hold. According to Fermat's Last Theorem, which has recently been proved as stated above, no such integers exist. The proof is beyond a math drooler. What is needed is a simpler proof that we all can enjoy. One is not given here. Other relation may be found that could lead to a simple proof. So the road is open for any traveler to try and transverse.

Math Drooler 2 July 2012