Fibonacci Numbers

A definition of Fibonacci numbers will be given, then an example. Let the n^{th} Fibonacci number F_n be defined by $F_n = F_{n-1} + F_{n-2}$ where $F_0 = 0$ and $F_1 = 1$.. The first 10 Fibonacci numbers can be easily derived

n	0	1	2	3	4	5	6	7	8	9
F_n	0	1	1	2	3	5	8	13	21	34

Consider F_n where n is an odd integer.

$$
\begin{aligned}
F_n &= F_{n-1} + F_{n-2} \\
&= (F_{n-2} + F_{n-3}) + F_{n-2} \\
&= 2F_{n-2} + F_{n-3} = F_3F_{n-2} + F_2F_{n-3} \\
&= 2(F_{n-3} + F_{n-4}) + F_{n-3} \\
&= 3F_{n-3} + 2F_{n-4} = F_4F_{n-3} + F_3F_{n-4} \\
&= 3(F_{n-4} + F_{n-5}) + 2F_{n-4} \\
&= 5F_{n-4} + 3F_{n-5} = F_5F_{n-4} + F_4F_{n-5} \\
&\vdots \\
&= F_mF_{n-(m-1)} + F_{m-1}F_{n-m}
\end{aligned}
$$

From construction a pattern begins to emerge. The coefficients are generated just as Fibonacci numbers are generated and are Fibonacci numbers. The result is

(1) $$F_n = F_mF_{n-m+1} + F_{m-1}F_{n-m} \quad where \;\; n \geq m$$

This can be rigorously proved, which I will leave to those of rigor.

Taking $m = n - m + 1$ gives $m = \frac{n+1}{2}$, $m - 1 = \frac{n-1}{2}$ and $n - m = \frac{n-1}{2}$. Substituting these into Equation 1 yields

(2) $$F_n = F^2_{\frac{n+1}{2}} + F^2_{\frac{n-1}{2}}$$

Any Fibonacci number with a odd index can be expressed as the sum of two consecutive Fibonacci numbers squared. Equation 2 can be written in terms of the index m taking $n = 2m - 1$, giving

(3)
$$F_{2m-1} = F_m^2 + F_{m-1}^2$$
$$F_m^2 + F_{m-1}^2 = F_{2m-1}$$

Any two consecutive Fibonacci numbers squared is a Fibonacci number.

Let n be an even positive integer. From the definition of Fibonacci number

$$
\begin{aligned}
F_{n+1} &= F_n + F_{n-1} \\
F_n &= F_{n+1} - F_{n-1} \\
&= \left(F_{\frac{n+2}{2}}^2 - F_{\frac{n}{2}}^2\right) - \left(F_{\frac{n}{2}}^2 - F_{\frac{n-2}{2}}^2\right) \quad from\ Equ\ 2 \\
&= F_{\frac{n}{2}+1}^2 - F_{\frac{n}{2}-1}^2 \\
&= \left(F_{\frac{n}{2}+1} + F_{\frac{n}{2}-1}\right)\left(F_{\frac{n}{2}+1} - F_{\frac{n}{2}-1}\right) \\
&= \left(F_{\frac{n}{2}+1} + F_{\frac{n}{2}-1}\right) F_{\frac{n}{2}}
\end{aligned}
$$

Two relations of interest are

(4)
$$F_n = F_{\frac{n}{2}+1}^2 - F_{\frac{n}{2}-1}^2$$
and

(5)
$$F_n = \left(F_{\frac{n}{2}+1} + F_{\frac{n}{2}-1}\right) F_{\frac{n}{2}}$$
where n is even.

These can be made to look more general by substituting $2n$ for n. Then

(4a)
$$
\begin{aligned}
F_{2n} &= F_{n+1}^2 - F_{n-1}^2 \\
&= (F_{n+1} + F_{n-1})(F_{n+1} - F_{n-1}) \\
&= (F_n + F_{n-1} + F_{n-1})F_n \quad definition\ F_n = F_{n-1} + F_{n-2}
\end{aligned}
$$

(5a)
$$F_{2n} = (2F_{n-1} + F_n)F_n$$

Before commenting on Equation 4a, Fibonacci numbers will be generalize as follows. Let the n^{th} general Fibonacci number $G_n(m,k)$ be defined by

$$G_n(m,k) = G_{n-1}(m,k) + G_{n-2}(m,k) \quad where \quad G_0(m,k) = m \quad and \quad G_1(m,k) = k$$

As an example, the first 10 general Fibonacci numbers of $G_n(3,7)$ are

n	0	1	2	3	4	5	6	7	8	9
$G_n(3,7)$	3	7	10	17	27	44	71	115	186	301

The general case can be reduced down to normal Fibonacci numbers.

n	$G_n(m,k)$	
0	m	m
1	k	$mF_0 + kF_1$
2	m+k	$mF_1 + kF_2$
3	m+2k	$mF_2 + kF_3$
4	2m+3k	$mF_3 + kF_4$
5	3m+5k	$mF_4 + kF_5$
6	5m+8k	$mF_5 + kF_6$
...	...	
n		$mF_{n-1} + kF_n$

The coefficients are added in a way that fits the definition of Fibonacci numbers and the general pattern is seem. I will leave the rigor to the rigorous. In general (only positive numbers are being considered)

(6) $$G_n(m,k) = mF_{n-1} + kF_n$$

From Equation (4a)

$$\begin{aligned} F_{2n} &= F_{n+1}^2 - F_{n-1}^2 \\ &= (F_n + F_{n-1})^2 - F_{n-1}^2 \\ &= F_n^2 + 2F_nF_{n-1} + F_{n-1}^2 - F_{n-1}^2 \\ F_{2n} &= F_n^2 + 2F_nF_{n-1} \end{aligned}$$

From Equation (6)

$$G_n(2,1) = 2F_{n-1} + F_n$$

$$\Longrightarrow F_{n-1} = \frac{G_n(2,1) - F_n}{2}$$

Substituting $$F_{2n} = F_n^2 + 2F_n\left(\frac{G_n(2,1)-F_n}{2}\right)$$
$$= F_n^2 + F_n G_n(2,1) - F_n^2$$
$$= G_n(2,1)F_n$$

(7) $$\boxed{F_{2n} = G_n(2,1)F_n}$$.

No even index Fibonacci number is a prime for which $G_n(2,1) \neq 1$. So all Fibonacci numbers of even index except index 2 are non-prime.

From the definition of general Fibonacci number

$$G_n = G_{n-1} + G_{n-2} = F_1 G_{n-1} + G_2 F_{n-2}$$
$$= (G_{n-2} + G_{n-3}) + G_{n-2}$$
$$= 2G_{n-2} + G_{n-3} = F_3 G_{n-2} + F_2 G_{n-3}$$
$$= 2(G_{n-3} + G_{n-4}) + G_{n-3}$$
$$= 3G_{n-3} + 2G_{n-4} = F_4 G_{n-3} + F_3 G_{n-4}$$
$$= 3(G_{n-4} + G_{n-5}) + 2G_{n-4}$$
$$= 5G_{n-4} + 3G_{n-5} = F_5 G_{n-4} = F_4 G_{n-5}$$
$$\vdots$$
$$= F_m G_{n-(m-1)} + F_{m-1} G_{n-m}$$

The m can be any integer greater than n. If n is odd, taking $m = \frac{n+1}{2}$ gives

(8) $$\boxed{G_n = F_{\frac{n+1}{2}} G_{\frac{n+1}{2}} + F_{\frac{n-1}{2}} G_{\frac{n-1}{2}}}$$.

If n is even, taking $m = \frac{n}{2}$ gives

(9) $$\boxed{G_n = F_{\frac{n}{2}} G_{\frac{n}{2}+1} + F_{\frac{n}{2}-1} G_{\frac{n}{2}}}$$

If $m = n$, one gets $G_n = F_n G_1 + F_{n-1} G_0$, which is the same as Equation 6.

Now, start with a large number and reduce it to the m and the k arguments of G_0. To make it clearer the first two terms of the series $G(m,k)$ are m and k. Let the last terms be q and p, ie. $G_n(m,k) = p$ and $G_{n-1}(m,k) = q$. Then define H such that

(10)
$$\boxed{H_0(p,q) = G_n(m,k)}$$

Then $H_n(p,q) = G_0(m,k)$. Start with some number p. Take some number $q < p$ and work down to the base numbers m and k in a general Fibonacci series and find the general Fibonacci number for p, i.e. $p = G_n(m,k)$.

From construction

n	0	1	2	3	4	5	...	n
$H_n(p,q)$	p	q	p-q	-p+2q	2p-3q	-3p+5q	...	$(-1)^n F_{n-1} p + (-1)^{n-1} F_n$

(11)
$$\boxed{H_n(p,q) = (-1)^n p F_{n-1} + (-1)^{n-1} q F_n \qquad where\ F_{-1} = 1}$$

The index n can continue to any large number. However, end indexing upward at the point $H_n(p,q) \leq H_{n+1}(p,q)$. Then

(12)
$$\boxed{\begin{gathered} H_n(p,q) = G_0(m,k) = m \\ H_{n-1}(p,q) = G_1(m,k) = k \\ k > m \end{gathered}}$$

To play with some numbers take $p = 12345$ and $q = 8000$. The series for H is

$$H(12345{,}8000) \quad \rightarrow \quad 12345 \quad 8000 \quad 4345 \quad 3655 \quad 690.$$

One ends at 690 since $690 < (3655 - 690)$. The terms go from $n = 0$ to $n = 4$. From the definition for H one has $H_0(12345{,}8000) = G_4(690{,}3655)$.

Taking $q = 7000$ gives

$$H(12345{,}7000) \quad \rightarrow \quad 12345 \quad 7000 \quad 5345 \quad 1655$$

and $H_0(12345{,}7000) = G_3(1655{,}5345)$.

What is of interest is the size of the series. For $q = 8000$ there are five members in the series. For $q = 7000$ there are four members. What value of q will yield the most members and give the most basic general Fibonacci number for p. This is where we need to use far more highly abstract reasoning than a math drooler is accustom too. The gut feeling is that $H(p,q)$ will have the most members when $q = \beta p$ where $\beta = \lim_{n\to\infty} \frac{F_n}{F_{n+1}} = 0.618034$. The value of q will need to be rounded to an integer. For the value of p above, using the gut feeling one has $q = 7629$ or 7630, depending on how $q = \beta p$ is rounded, giving

$$H(12345{,}7629) = 12345\ \ 7629\ \ 4716\ \ 2913\ \ 1803\ \ 1110\ \ 693\ \ 417\ \ 276\ \ 141\ \ 135\ \ 6 = G_{11}(6{,}135)$$

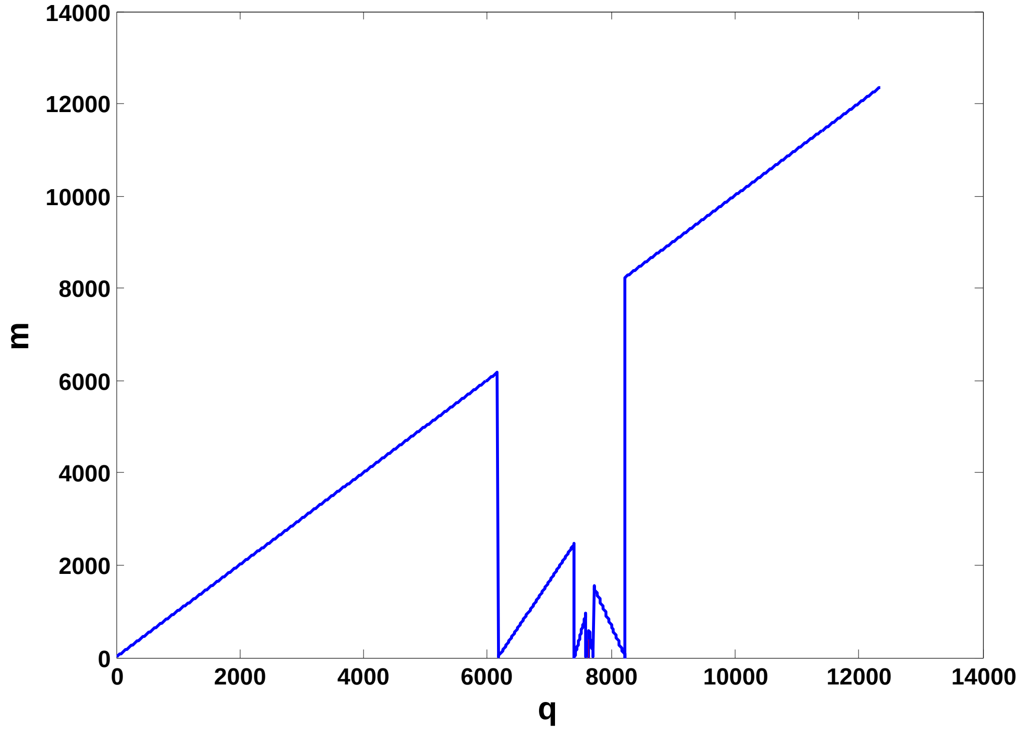
and

$$H(12345{,}7630) = 12345\ \ 7630\ \ 4715\ \ 2915\ \ 1800\ \ 1115\ \ 685\ \ 430\ \ 255\ \ 175\ \ 80 = G_{10}(80{,}175)$$

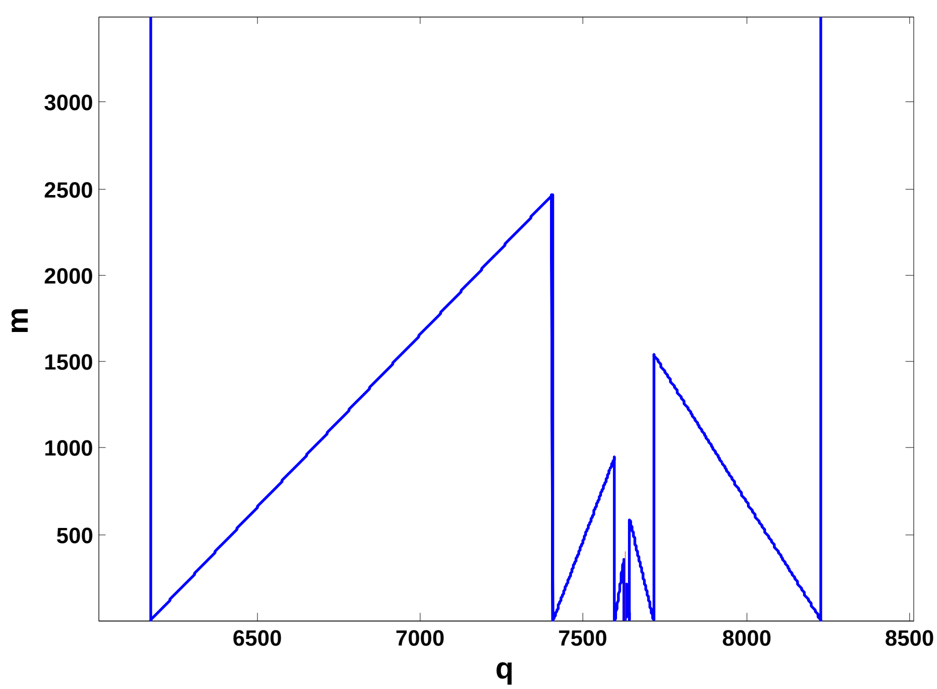
There are 12 numbers in the series for $H(12345{,}7629)$. The series $H(12345{,}7630)$ has 11 terms. Is there any q for which $H(12345, q)$ has more than 12 terms? The answer is no. So $q = \beta p$ gives the most basic, the series with the most terms, general Fibonacci number for $p = 12345$. Is the gut feeling true in general? That is not known.

Another gut feeling is that $\lim_{n\to\infty} \frac{G_{n-1}(m,k)}{G_n(m,k)} = \beta \quad \forall \quad (m,k)$. This is easy to prove using Equation 6.

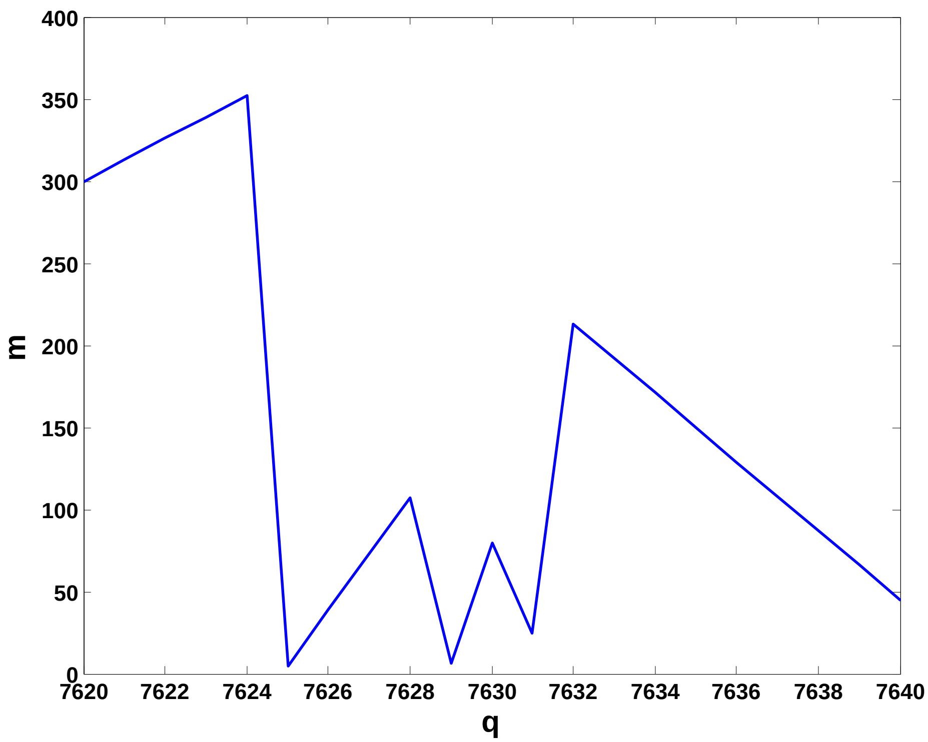
Now that there are some implements to navigate with, it is time to walk the landscape. The landscape is wide with many paths. One path is sufficient for today. The hiking boots used will be Equation 12. A value will be given for p, the value for q will vary and the corresponding values for m will be plotted. Let $p = 12345$ and let $q = 1\ to\ p$. Plotting m versus q gives the following plot.



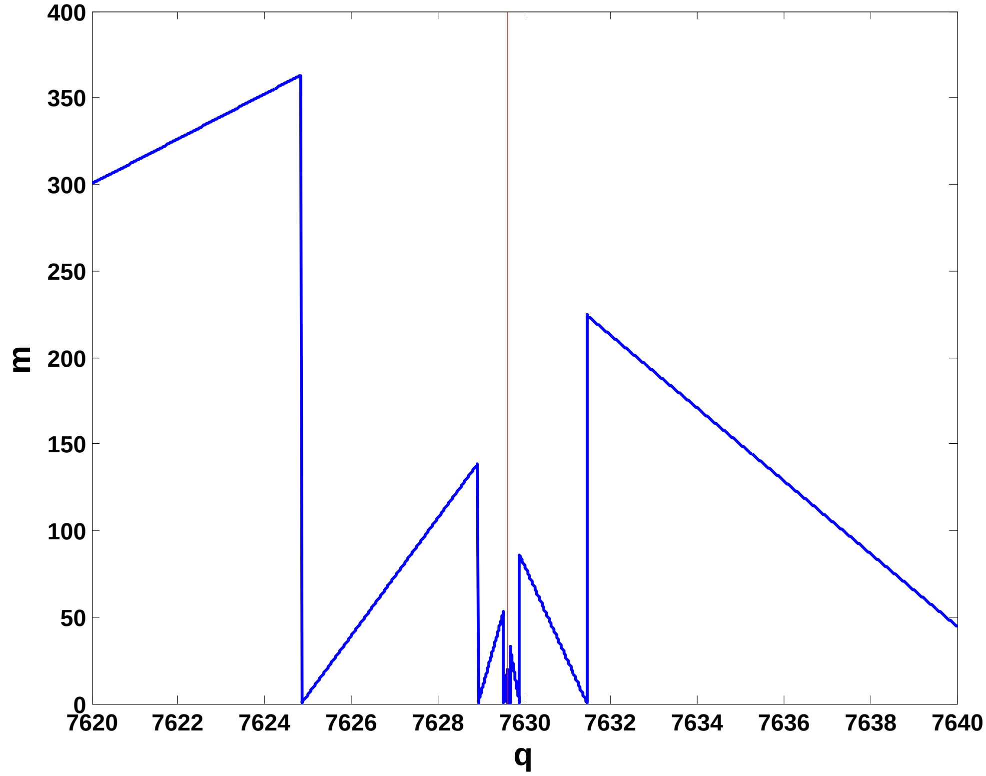
Expanding the plot gives



As the plot is expanded it looks self similar. Greater expansion gives the following.



The self similar nature ends on extreme magnification. However, it must be kept in mind that only integers are being considered, i.e. q varies in integer values. The same computations can be performed with q varying in real numbers.



For real q the curve is self similar about some point magnified as much as one likes. There is placed a red line at that point. No matter how fine an area examined it will look the same. How does one know exactly where that point is? Well, it is at βp where β is given above, $\beta = \lim_{n\to\infty} \frac{F_n}{F_{n+1}}$. In the present case with $p = 12345$ the line about which the curve is self similar is $q = \beta p = \beta \times 12345 = 7629.6$, which looks in the ball park in the plot above.

This is one path traveled on the mathematical landscape of Fibonacci numbers. Many other paths can be taken, but not now. It is time to move on to other hills.

Math Drooler 1 Jan 2009
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