Fractal Fun

Fractals can put our desktop number laboratory to its proper use. One needs only pick a relation, do copious amounts of feedback and display the results. The terrain is not too difficult to hike and the scenery can be beautiful. However, only a glimpse will be seen here. To get a fuller and richer sight one need take the best trail in the world of fractals and chaos, which is the two volume set "Fractals For the Classroom" by Pettgen, Jurgens and Saupe.

The first step is to pick a feedback relation of the type

$$x_{n+1} = f(x_n).$$

Starting with some x_0 use the relation above to compute successive values for x_n. For our purpose the function f should be simple but not so simple that nothing interesting happens. The simplest function that will give interesting results is

$$f(x) = a(x^2 - 1).$$

The factor a can be any number. The feedback process will be

(1) $$x_{n+1} \leftarrow a(x_n^2 - 1).$$

Choose some starting point x_0 and some factor a and observe what develops. If $x_0 = \pm 1$ then clearly nothing interesting is going to happen regardless of a. Figure 1 shows the result of the first 100 iterations for $a = 0.5$ and $x = 0.5$.

Figure 1   (a=0.5, x=0.5)
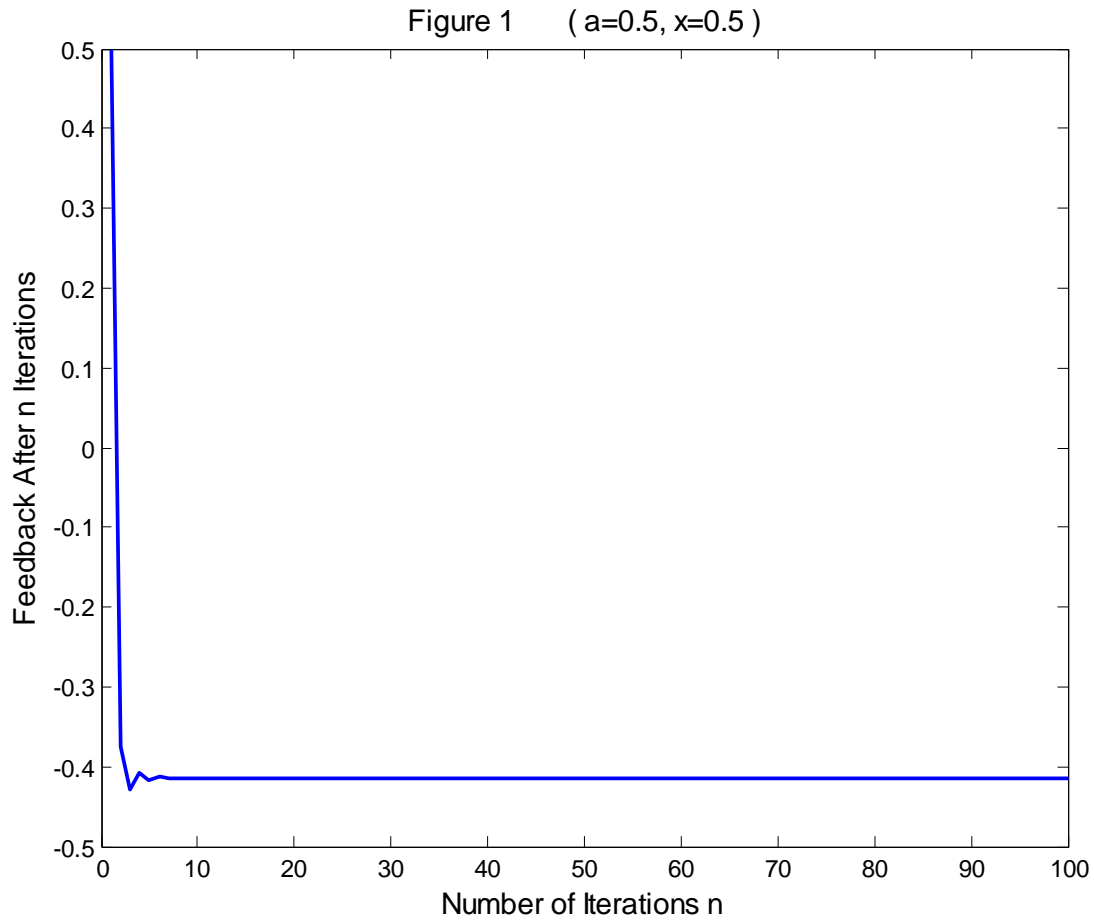
The value of the feedback process quickly approaches a constant value. It overshoots this value and oscillates about it settling in after a few iterations. For the next test take $a = 0.8$ and $x = 0.5$. The results are shown in Figure 2.

Figure 2   (a=0.8, x=0.5)
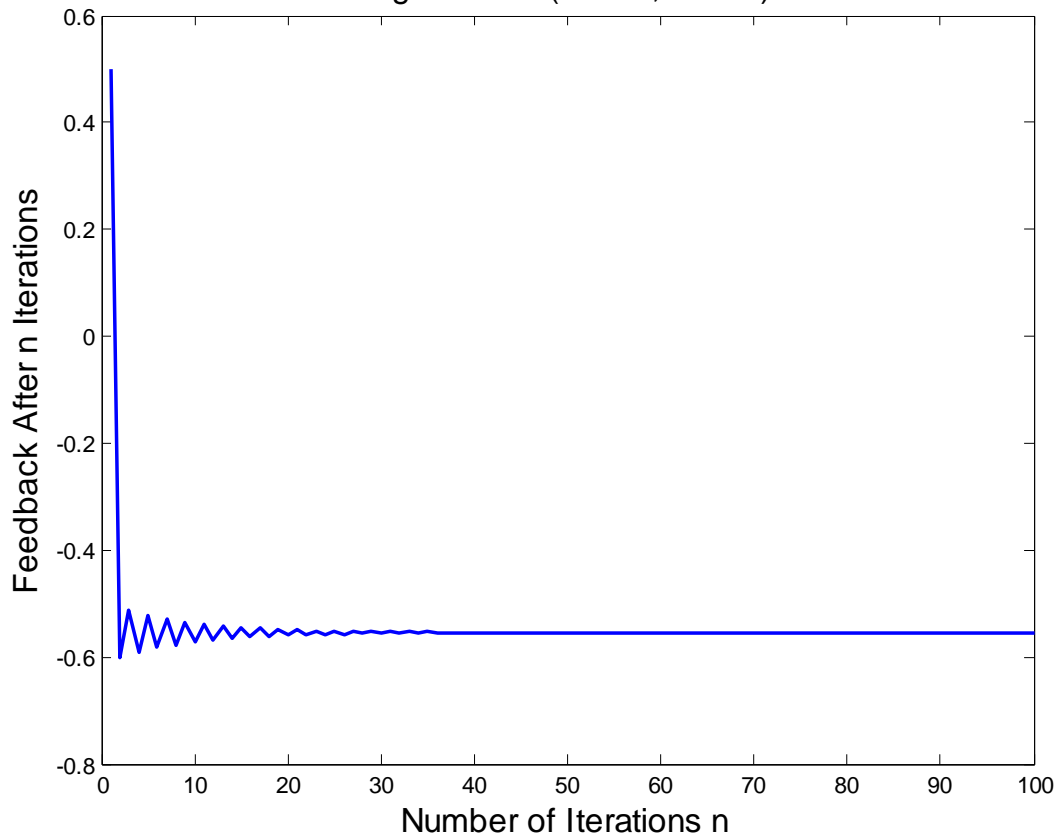
The behavior is the same as in Figure 1 except it takes longer for the oscillations to die out. Next look at $a = 1$ and $x = 0.5$. The results are shown in Figure 3.

Figure 3　　(a=1.1, x=0.5)
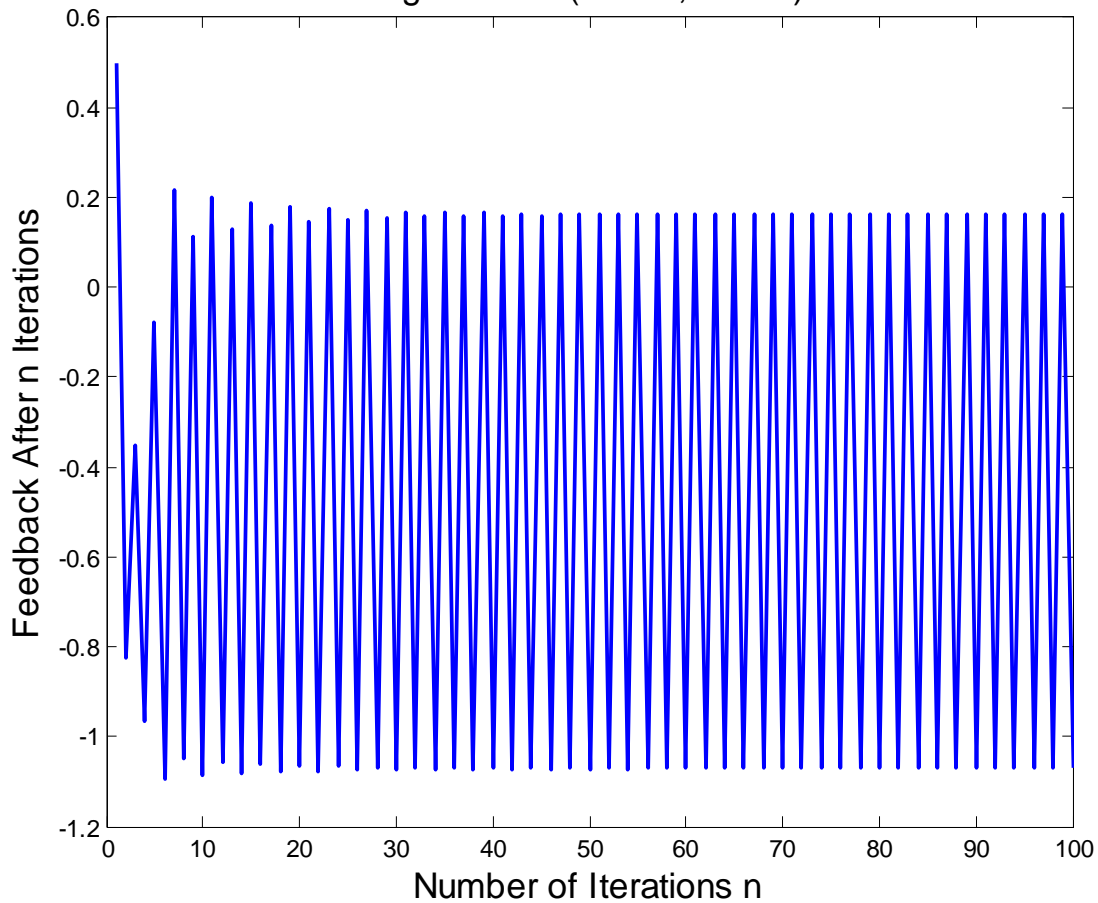
The behavior here is a little different. The values settle down to an oscillation between two values. Taking the number of iterations further, say to 1000, does not change things. The process has settled in, at least on the current number crunching machine. As pointed out in "Fractals for the Classroom" the machine the calculations are made on can affect the outcome in feedback processes. That will be ignored here, the machine being a real world entity within the feedback process. It is a variable maybe, but a variable of one unit for the discussion here.

Moving on let $a = 1.2$ and $x = 0.5$ and take 200 iterations. The values in Figure 4 look as though they may repeat at some point. However, that is not what is found. Taking the iteration out 100,000 times, the $100,000^{th}$ value will not be found in any preceding values. The behavior is chaotic.

Figure 4   (a=1.2, x=0.5)
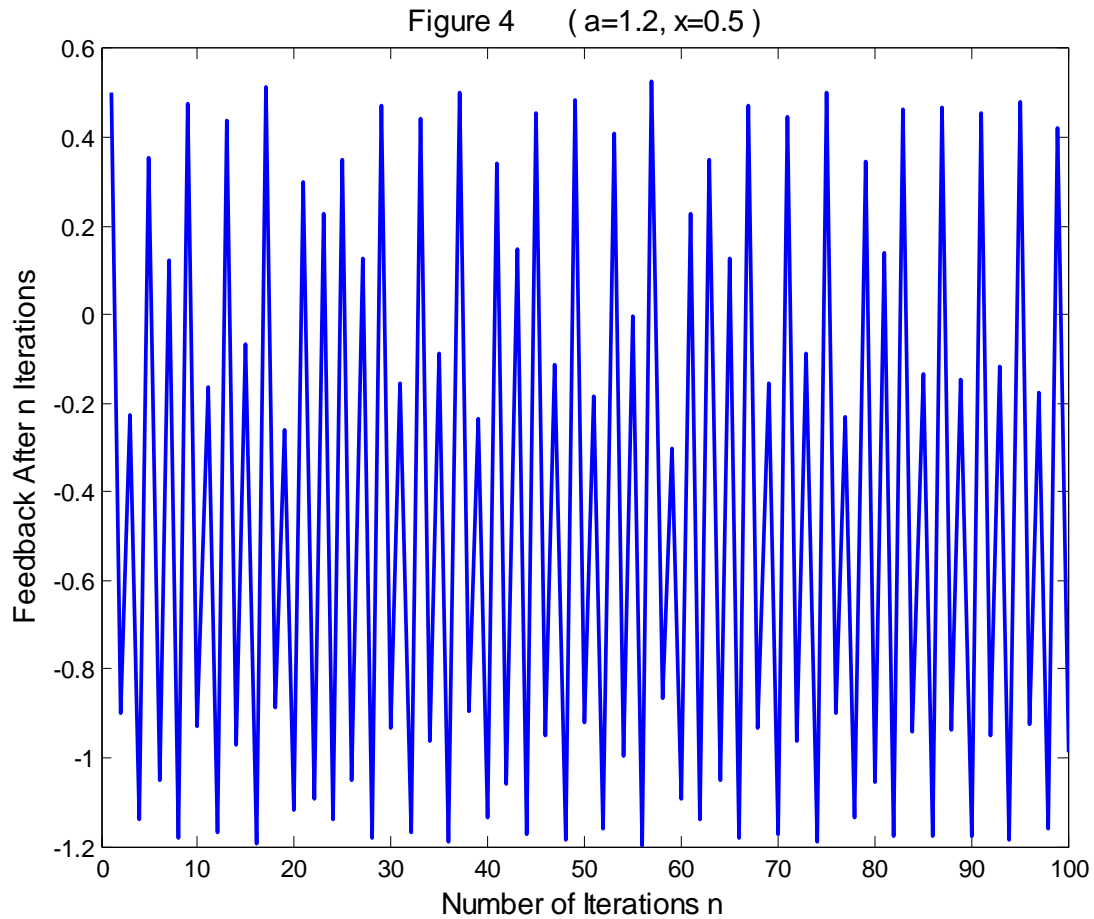
Three types of behavior have been displayed. There is a forth type depict in Figure 5. The value of a is the square root or 2 plus a little more, $a = \sqrt{2} + 0.001$. The value of the iterations marches off to infinity and is off the scale before completing 35 iterations.

Figure 5   (a=sqrt(2)+0.001, x=0.5)
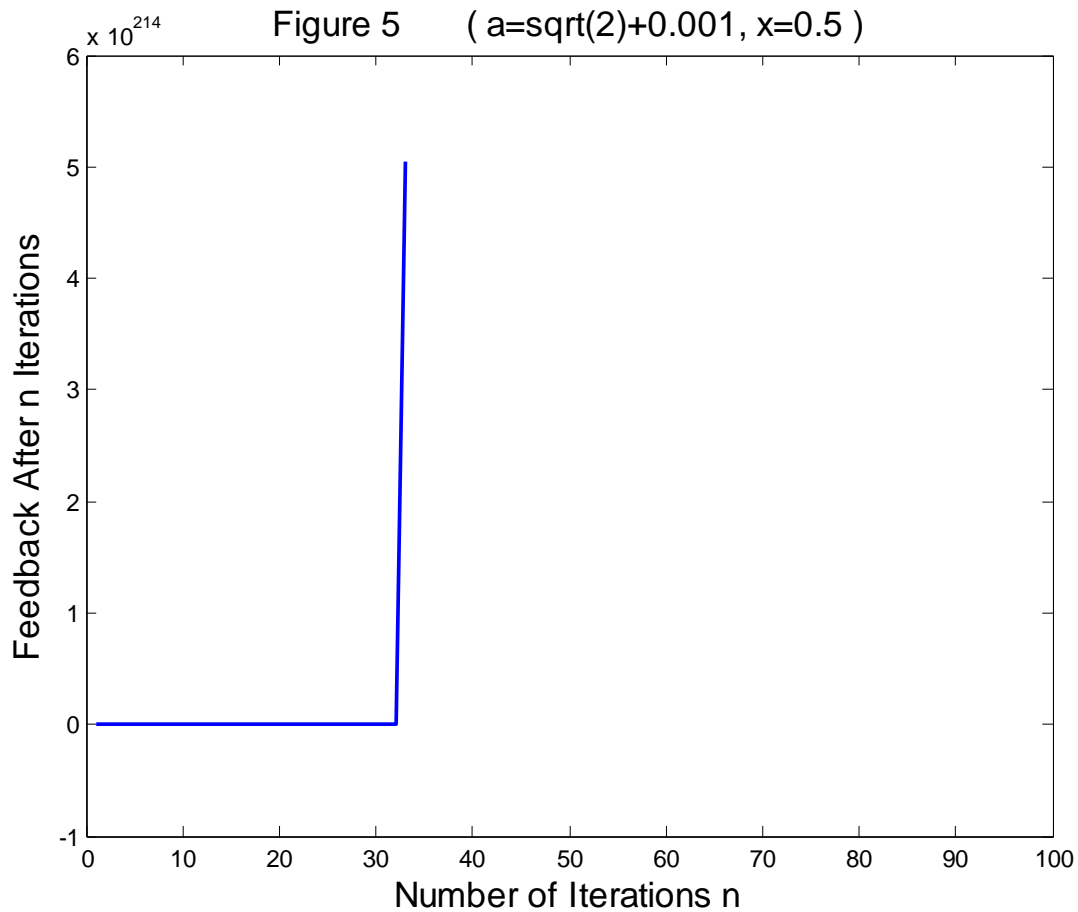
These line plots are very interesting but no one ever said that a line is worth a thousand words. Two dimensional pictures are more appealing to the eye. An image could be made by varying a over $-\sqrt{2}$ to $+\sqrt{2}$ and x over some range. A criterion has to be determined to give some idea of the behavior of the iteration. Below are plots of iterations for $x = 0$ and various values for a. Next to each plot is a plot of the absolute value of the FFT for iterations 101 to 1124. At 101 the iteration calms down from its starting point and ending at 1124 gives 2^{10} iterations, which is convenient for the FFT.
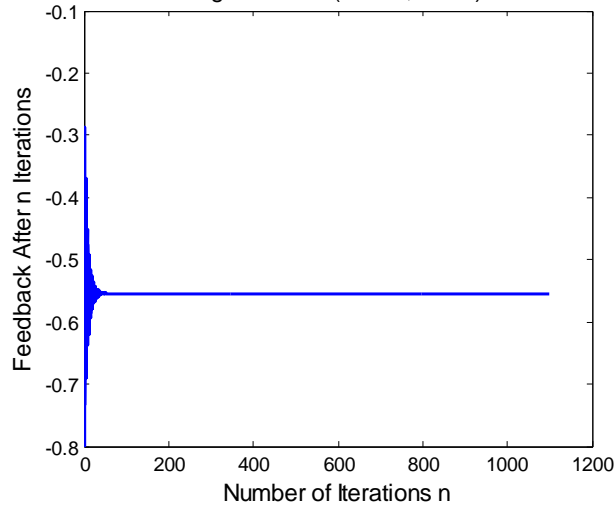
Figure 6a   (a=0.8, x0=0)

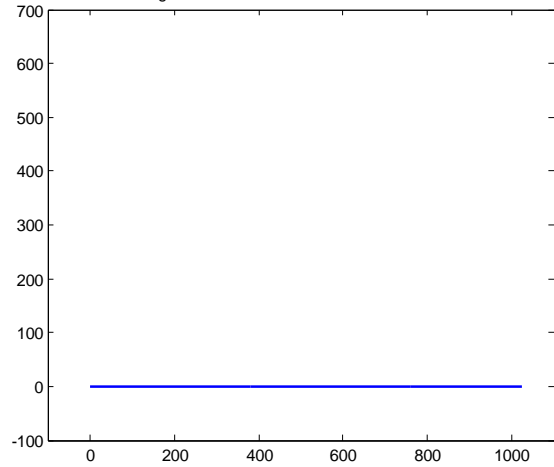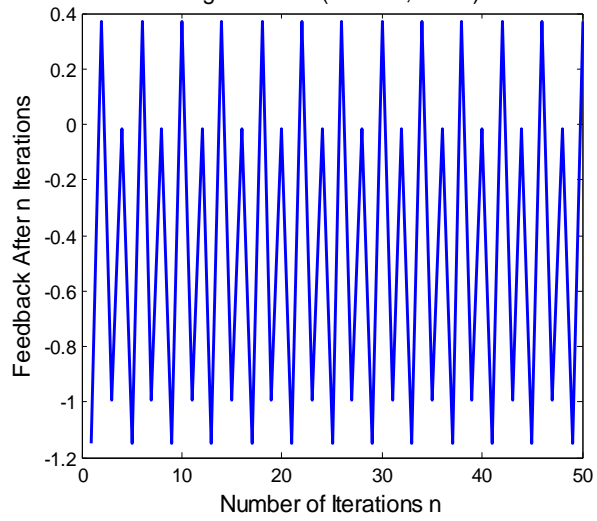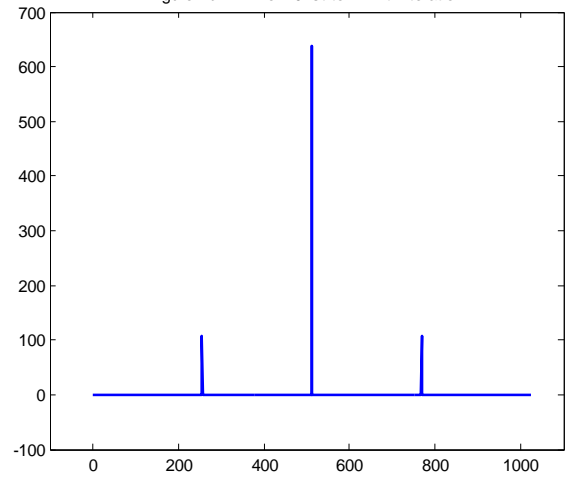Figure 6b   FFT of 101st to 1124th iteration

Figure 7a   (a=1.15, x0=0)

Figure 7b   FFT of 101st to 1124th iteration

Figure 8a (a=1.2, x0=0)

Figure 8b   FFT of 101st to 1124th iteration

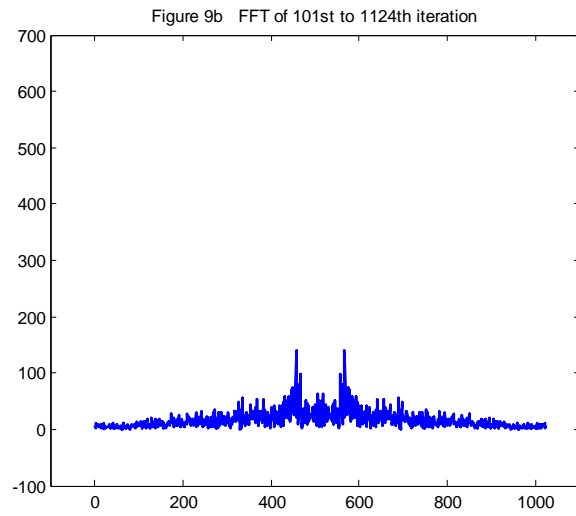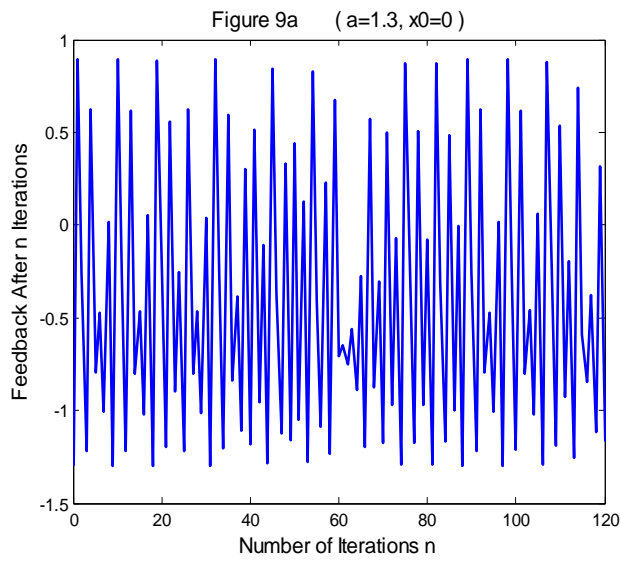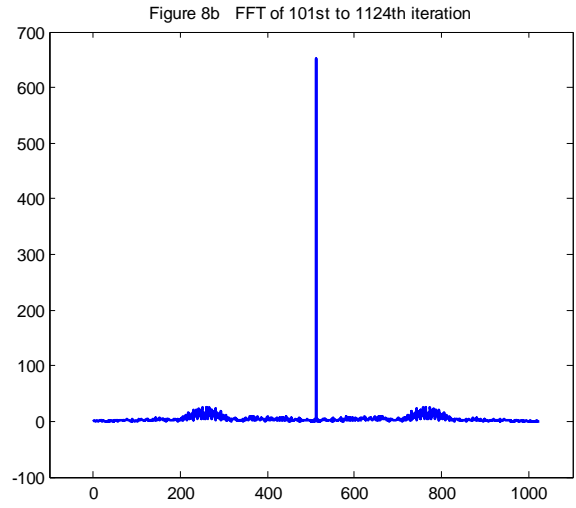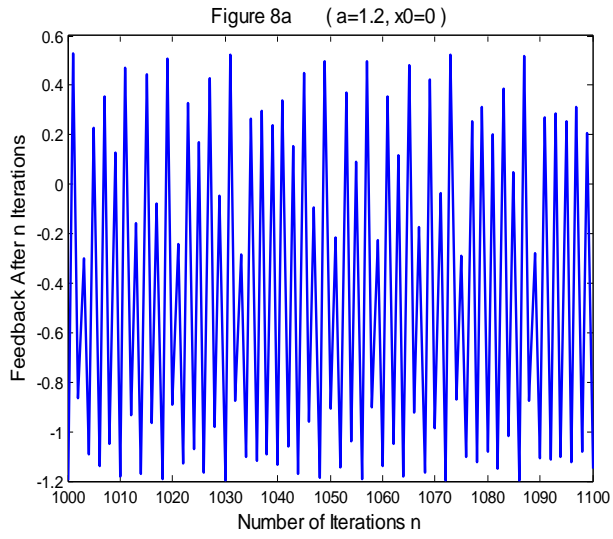Figure 9a (a=1.3, x0=0)

Figure 9b   FFT of 101st to 1124th iteration

Looking at the plots ones sees that as the iterations become less periodic the absolute value of the FFT becomes more scattered. The FFT is symmetric about its middle. Neglect the first point, which is not plotted, and the middle point. Sum the points from 1 to 511 and determine any possible relation. The table below is such a sum for each value of a in the plots above.

| a	FFT sum
0.8	0.0011
1.15	106.9
1.2	2237
1.3	8641

There are few data points, but it appears that the less periodic the iterations the greater the sum of the FFT. Lacking anything else simple to determine the behavior of the iterations this will be used and will be referred to as the FFT sum. Figure 10 shows the FFT sum for $x_0 = 0$ over a range of a.

Figure 10   (x0=0)
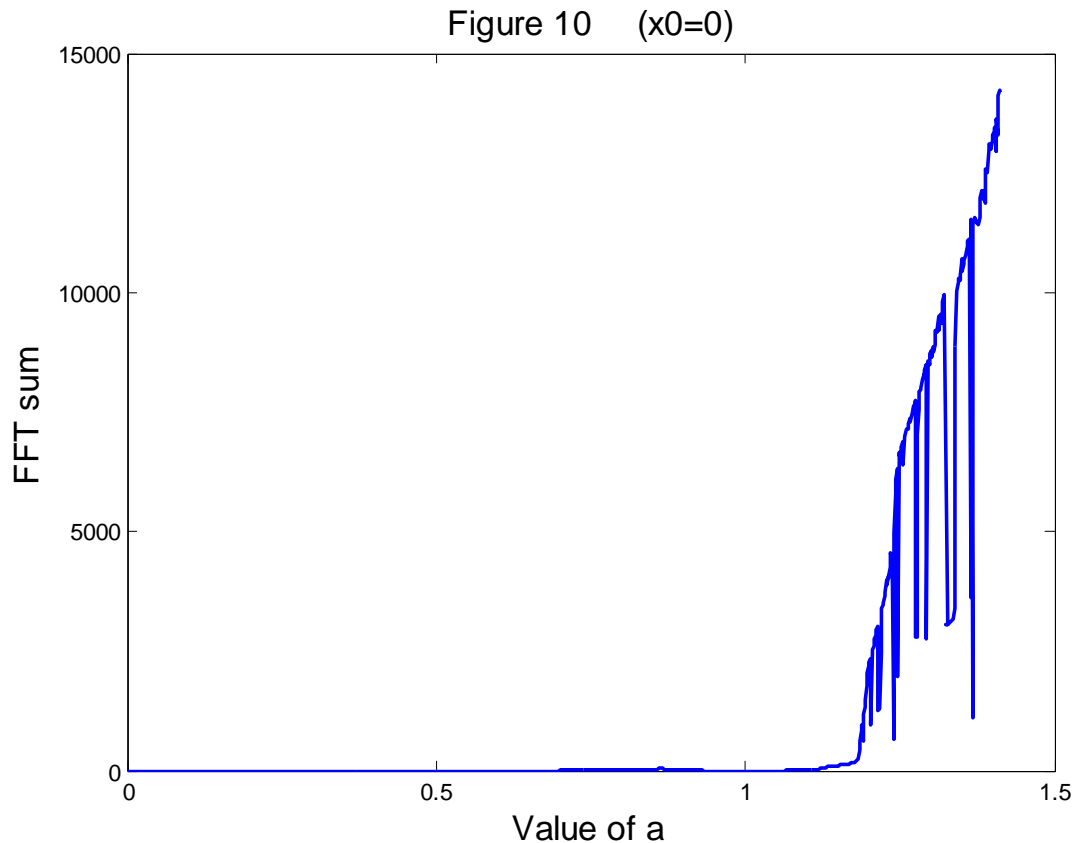
The initial x in the figure above was zero. The initial x can be varied as well as the value of a and a more interesting figure can be obtained. Figure 11 is such figure. The white area is indefinite expansion. The black area is flat line behavior. The gray areas are periodic or chaotic. Darker to lighter gray is simplest periodic to most complex chaotic, if there really is chaotic behavior. That is said because one does not know if the iterations will start to repeat just beyond the number of iterations being looked at, or if they will repeat beyond the ability of the machine being used.
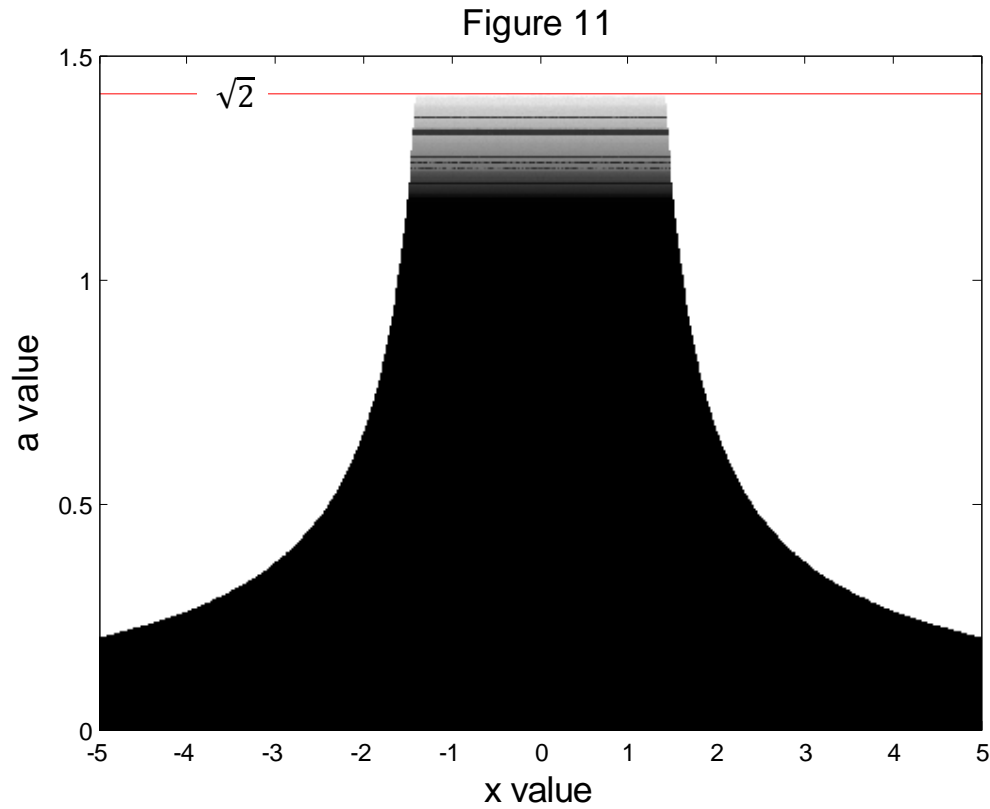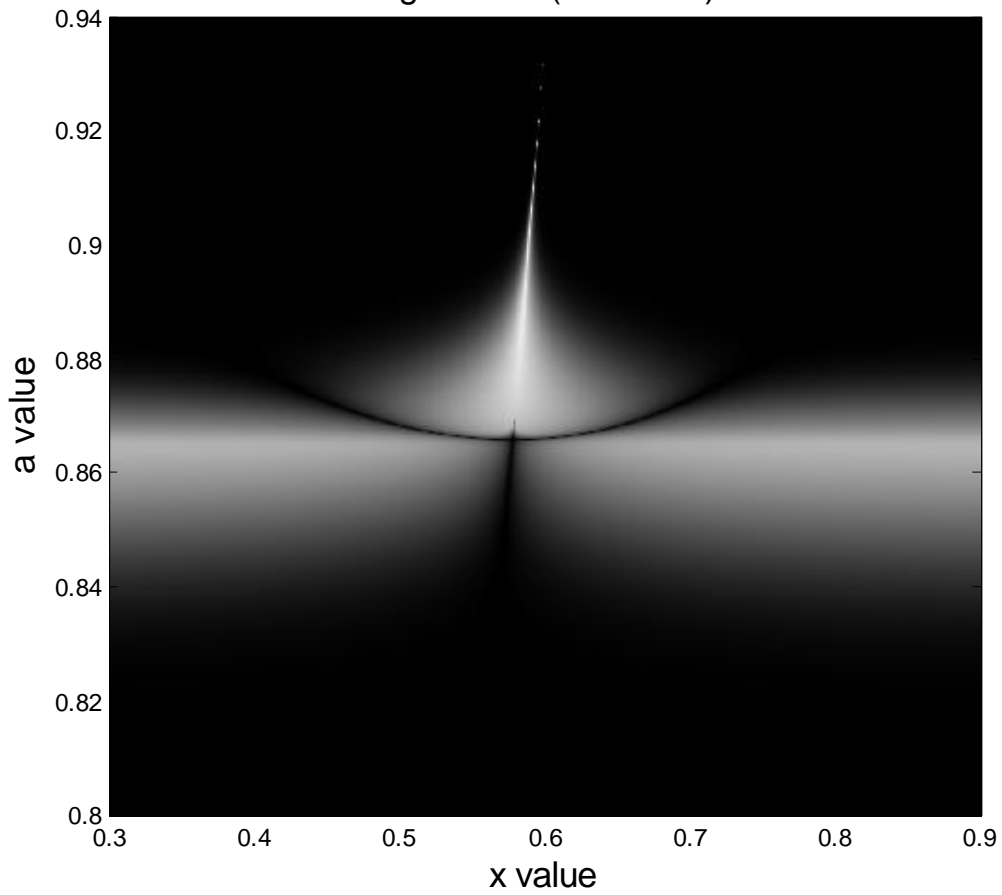
Figure 11

In Figure 10 it is seen that the FFT sum varies from 0 to 14,000. With such a variance and a steep incline toward the end of the plot, a log scale may be more instructive. To account for the problem of taking the log of 0, 1 will be added to the numbers comprising Figure 11. Figure 12 is the result.

Figure 12   (FFT sum)
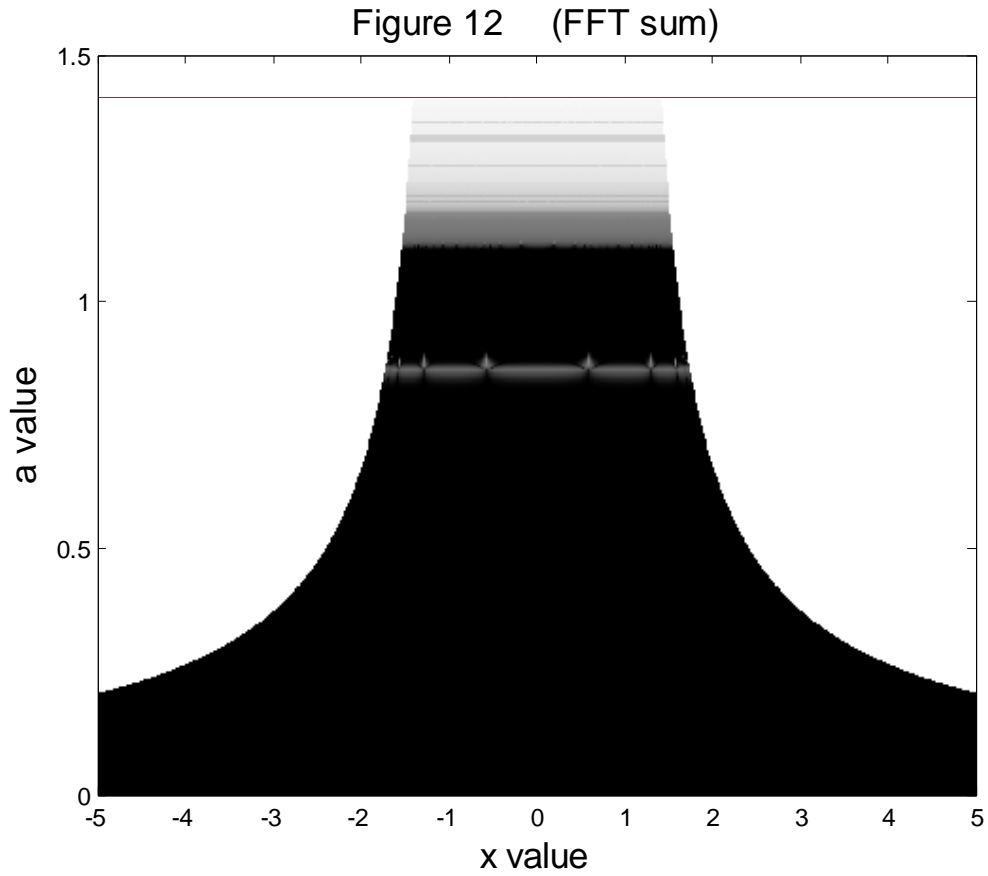
The shape in generally the same as in Figure 11, as it should be, but some information is better highlighted. Two areas that would be interesting to see expanded are the area near $x = 0.6$ and $a = 0.8$, and the area near $x = 1$ and $a = 1.3$. Figure 13 is an expansion near $x = 0.6$ and $a = 0.8$.

Figure 13   (FFT sum)

Transitions between areas that are well behaved (black) and those that are more chaotic (lighter gray) appear smooth and the figure is quite pleasing if not odd. A further expansion of the middle yields Figure 14.

Figure 14   (FFT sum)
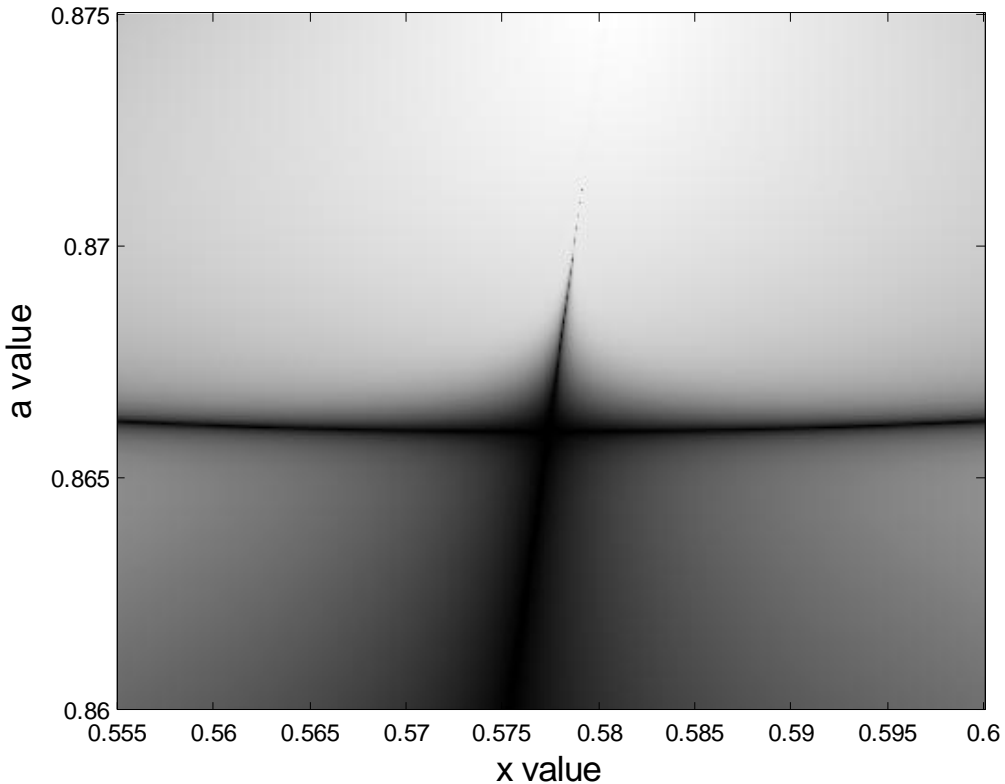
Little more is displayed. These are not fractals. To get a fractal one has to venture into the complex plane. The iteration is with complex numbers, Equation 1 becoming

$$(2) \qquad\qquad z_{n+1} \leftarrow a(z_n^2 - 1).$$

Without going into further detail a movie will shown the life and death of a fractal. As time goes on the value of a will vary from $0.6\ to\ 2.0$. Where color exists (i.e. within the fractal) for each a the iterations become similar sequences except for phase between adjacent lobes of the fractal except where chaotic behavior exists. To bring out structure in the fractal, the FFT sum is for iterations from the 11th to the 266th. This gives an FFT of 256 (2^8) points, which works out well. Further enhancements are made to bring out structure. At the first division or breakup of the fractal the iteration series goes from flat to oscillating. At subsequent divisions, the oscillations become of longer period or become chaotic. The gray area is where the iteration increases indefinitely, the darker the gray the faster the increase.

This will be called the void. There is some structure in the void that is brought out. Click on movie to observe. The quality had to be reduced to make the file size manageable for this low budget system. It may not play on older Windows Media players.

In Equation (2) the constant term a is a real number. It can be a complex number. If the complex part is positive then the fractal is given what appears as a counterclockwise twist to each lobe. A negative complex part results in opposite behavior. The twist is not about a common axis. Each lobe of the fractal appears to have its own axis. There is an infinite number of lobes. So there would appear to be an infinite number of axes. Perhaps one of those imagined rolled up dimension string people talk about. Play with fractal program to observe. Set the value for the real and imaginary parts of "a". Hit "Run" and wait. One can also go to the "a" plane and set the real and imaginary parts of "z". No further explanation of how to use the program is given.

Math Drooler   06 Dec 2011