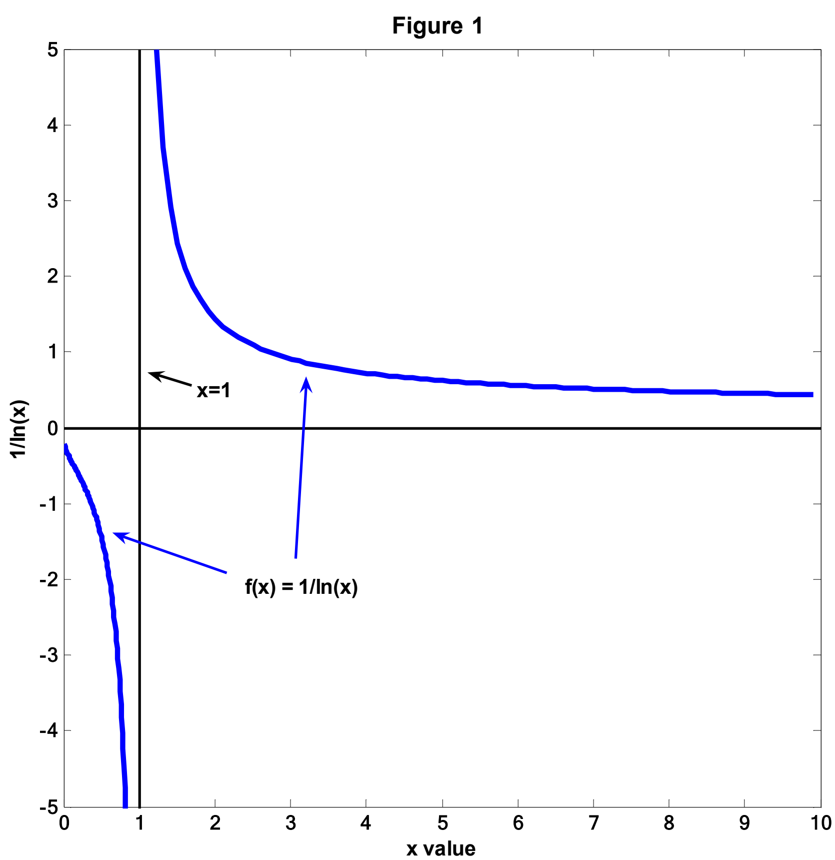
prime number fun

In the book "Prime Obsession" one comes across the log integral function and its relationship to prime numbers. The log integral function is defined by

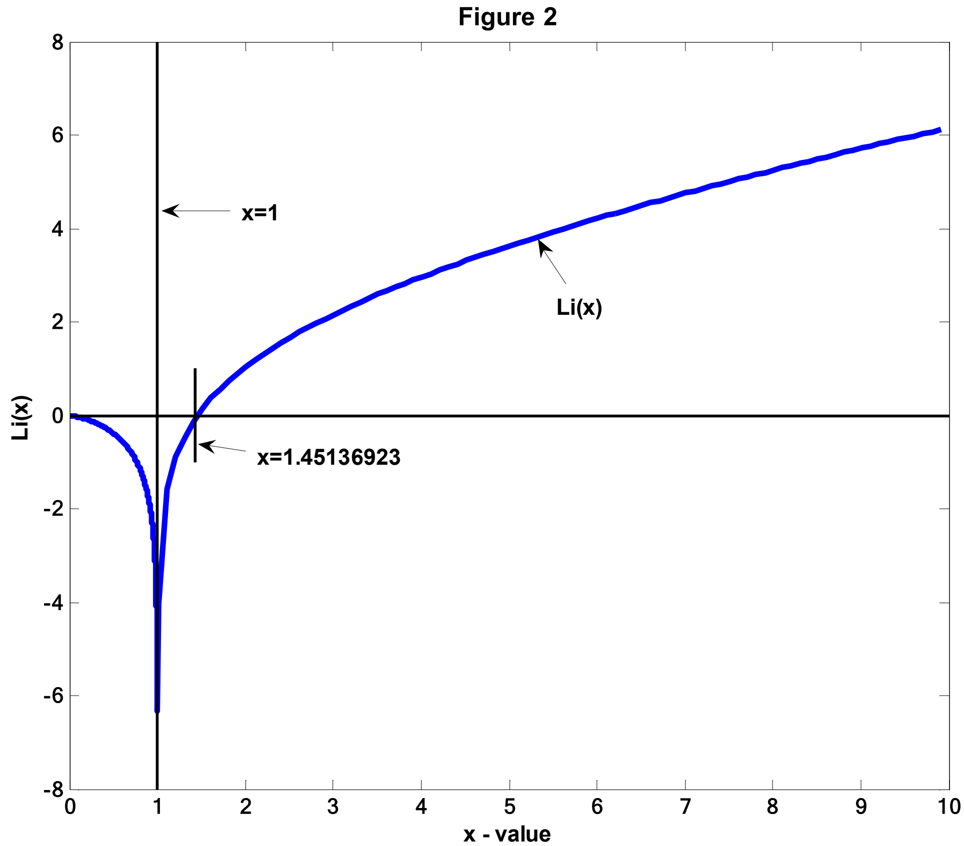
(1) $$Li(x) = \int_0^x \frac{1}{ln(t)} dt \; .$$

The function $f(x) = \frac{1}{\ln(x)}$ goes to $-\infty$ as x approaches $x = 1$ from the left and goes to $+\infty$ as x approaches $x = 1$ from the right. To integrate $f(x) = \frac{1}{\ln(x)}$ across $x = 1$ is difficult. Figure 1 shows a plot of $f(x) = \frac{1}{\ln(x)}$.



The difficulty in integrating over $x = 1$ is obvious. Integration is finding the area under the curve. The function going to $-\infty$ on one side of $x = 1$ and to

$+\infty$ on the other makes things messy. Others have done the messy work the results of which are presented in Figure 2.



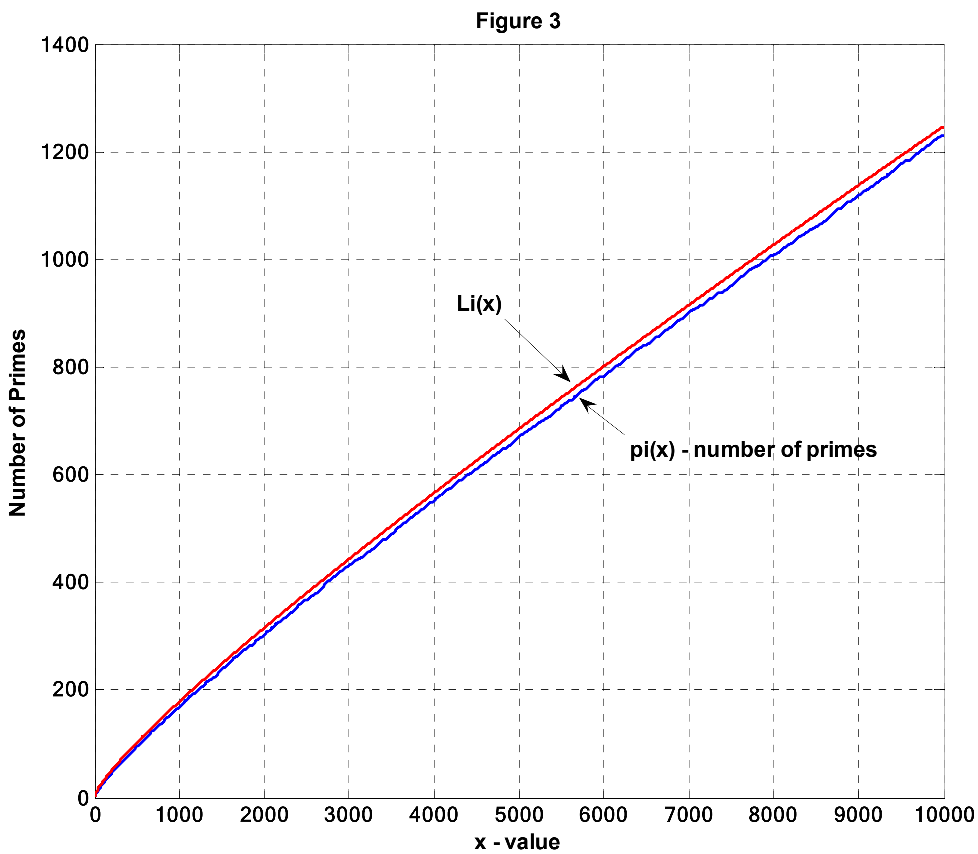
The function $Li(x)$ crosses the $x-axis$ at $x_0 = 1.45136923$. Taking the limits in Equation 1 to be $0\ to\ x_0$ the integral gives 0. It can be concluded that for $x_0 > 1.45136923$

(2) $$Li(x) = \int_0^x \frac{1}{ln(t)} dt = \int_{x_0}^x \frac{1}{ln(t)} dt.$$

The number of prime numbers up to and including n is denoted by $\pi(n)$. The prime number theorem states

(3) $$\pi(x) \sim Li(x).$$

Using Equation 2, the players in the prime number theorem can be plotted as in Figure 3.

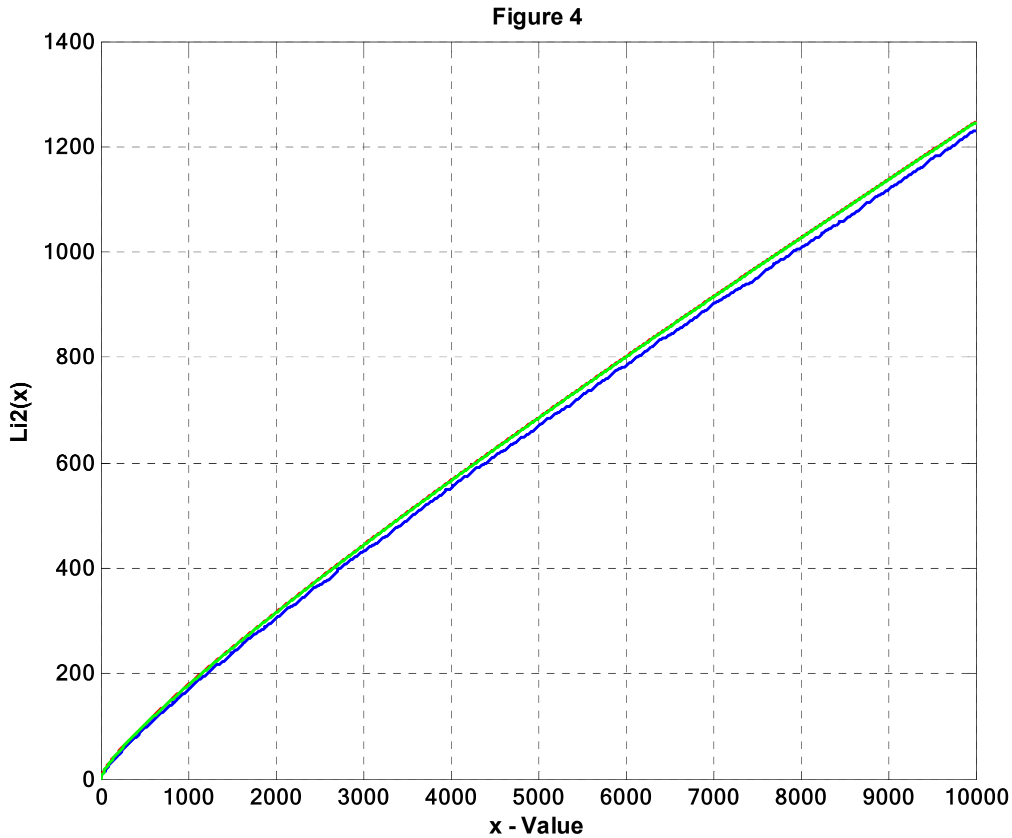


The two curves are quite close. Having a continuous function so well approximate the digital prime function, $\pi(x)$, is quite amazing. However, there is gap between the two where $Li(x)$ is always somewhat greater than $\pi(x)$. This is somewhat unsettling. There are several finagles that can be tried to remedy this to make it more pleasing.

Equation 2 shows that the integral can be taken from $x = x_0$. What if one does not worry about the singularity at $x = 1$ and integrates from $x = 2$? Define a modified log integral function as follows

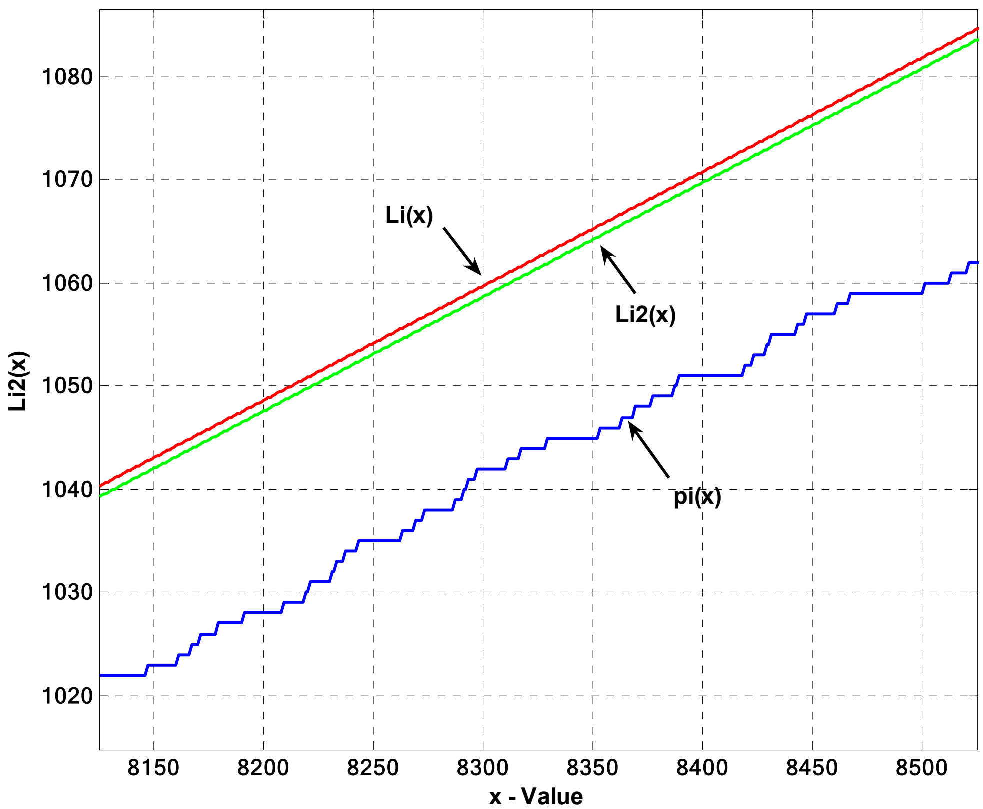
(4) $$Li_2(x) = \int_2^x \frac{1}{ln(t)} dt.$$

Figure 4 plots Equation 4 over the curves shown in Figure 3.



No difference is discernable between $Li(x)$ and $Li_2(x)$. Figure 5 is an expanded area of Figure 4.

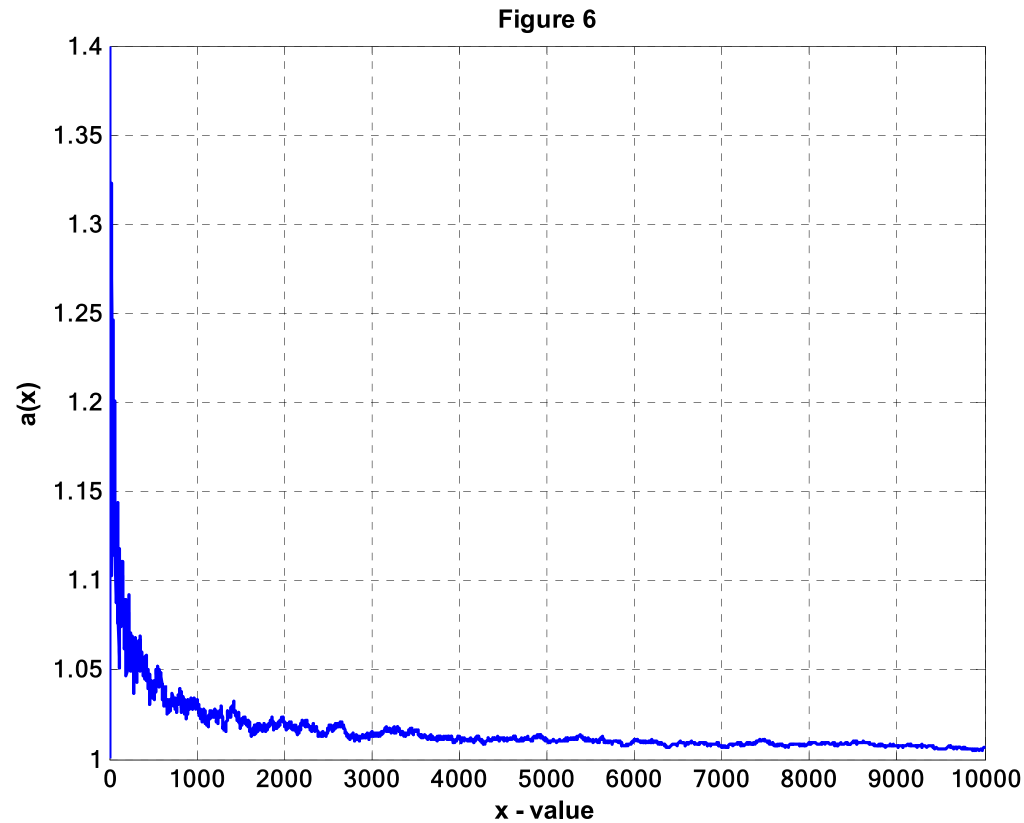
Figure 5



There is a small difference in the two Li functions. Li_2 is a slightly better fit to $\pi(x)$. One may look at integrating from ever larger x_0 to see if a more perfect fit can be found. But that is not the trail that will be taking here. Instead, define a function by

(5) $$Lia(x) = \int_2^x \frac{1}{\ln(x)^{a(x)}} dt.$$

The goal is to find $a(x)$. The method will be brute force. For different values of x, different values of $a(x)$ will be tried until $Lia(x) \cong \pi(x)$. Calculating $a(x)$ to four places gives results shown in Figure 6.

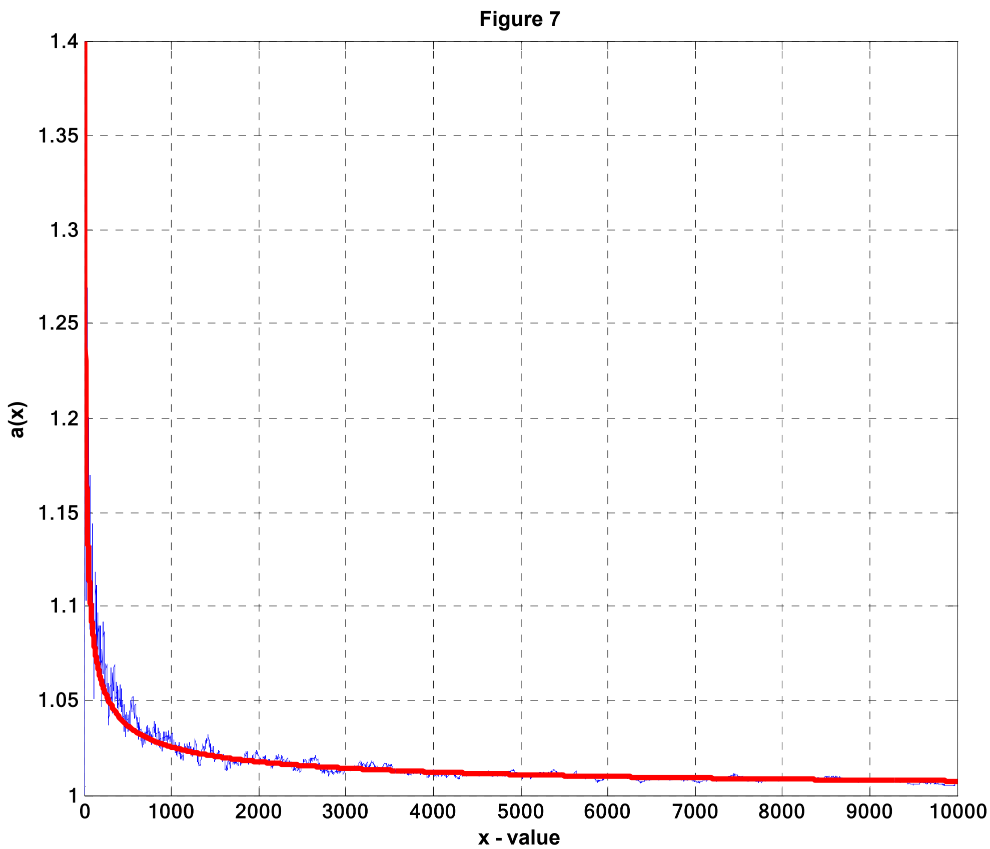


The curve $a(x)$ shows a general shape of an inverse of x approaching $y = 1$. This invites one to investigate the curve $a(x) = 1 + \frac{1}{x^b}$ to see if a general fit to $a(x)$ can be found. The term $\beta(x)$ can be an eyeball estimate of $a(x)$ from Figure 6. Solving for b gives

(6) $$b = -\frac{\ln(\beta(x)-1)}{\ln(x)} .$$

The table lists a few values for b using the eyeball average from Figure 6.

x	A(x)	b
1000	1.024	0.5399
2000	1.019	0.5214
4000	1.012	0.5333
8000	1.009	0.5241



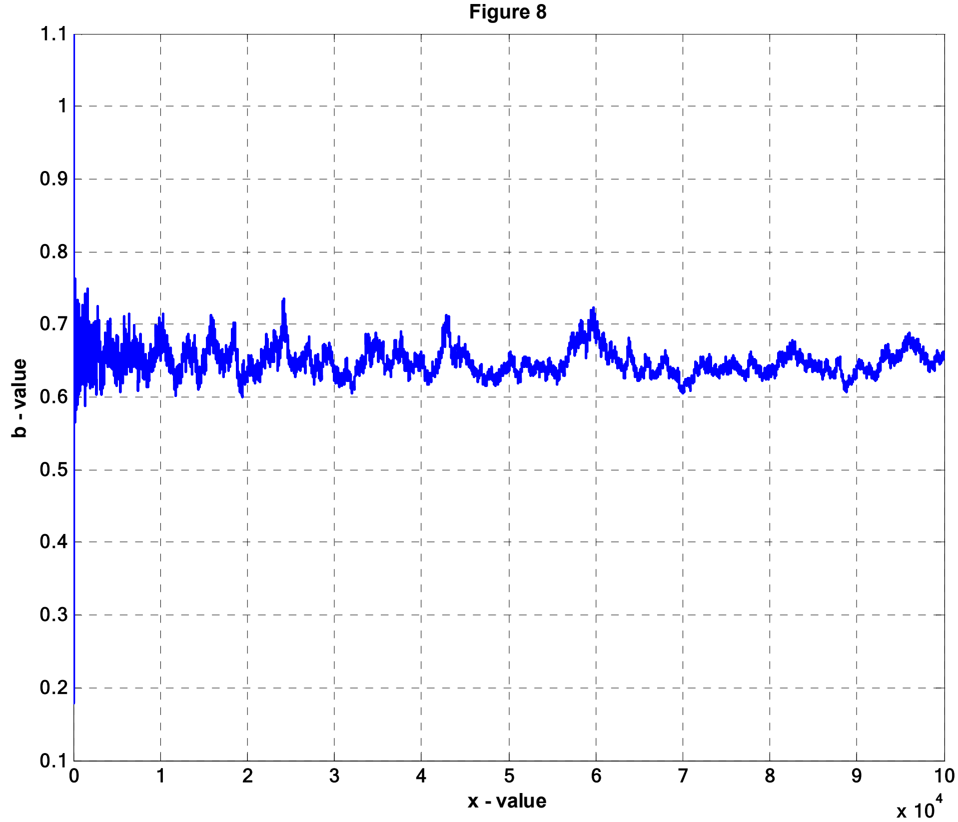
Taking $b = 0.53$, $\beta(x)$ is a good general fit to $a(x)$ as seen in Figure 7.

There is a subtlety one must be careful of. Integrals were taken to determine $a(x)$. The function $a(x)$ was taken as a constant over each integral. However, it was found that $a(x)$ was not a constant. With this consideration rewrite Equation 5 as follows.

(7) $$Lia(x) = \int_2^x \frac{1}{\ln(x)^{\left(1+\frac{1}{x^b}\right)}} dt.$$

Again use brute force as was use to find $a(x)$ to find b. The results of doing so are shown in Figure 8.

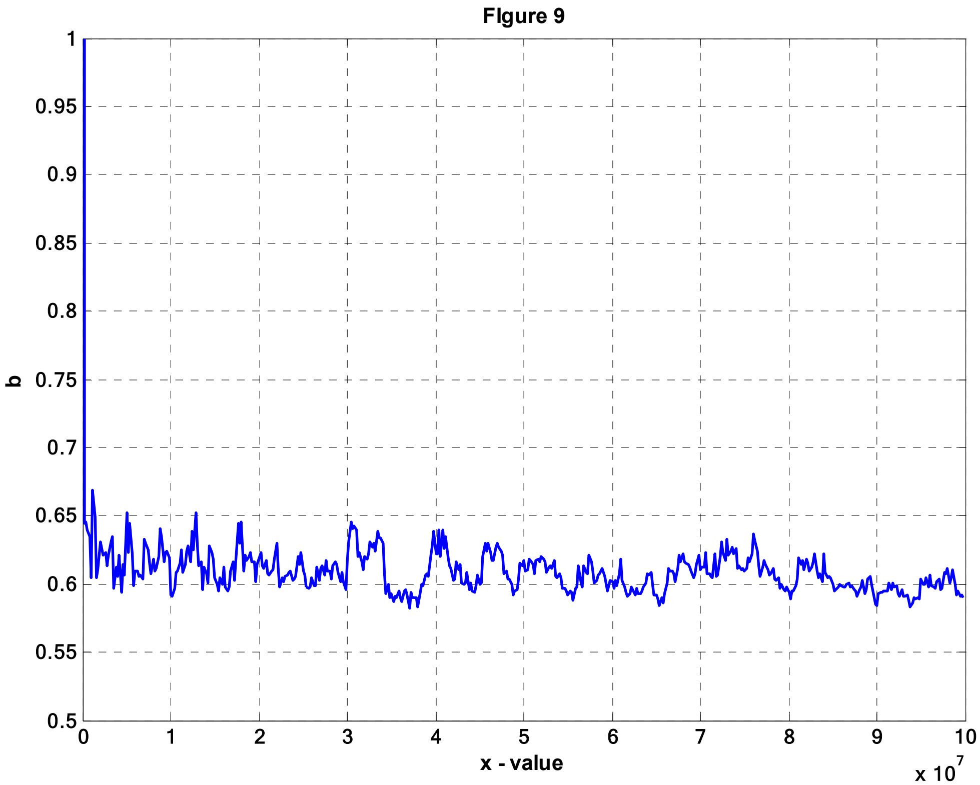
Figure 8



A good estimate of b is $b = 0.65$. This can be refined by making direct comparisons of $Lia(x)\ to\ \pi(x)$. In particular one can look at

(8) $$\beta(x) = Lia(x) - \pi(x)$$

for various values of b such that $\beta(x) = 0$. Figure 9 shows the results up to $x = 100{,}000{,}000$.



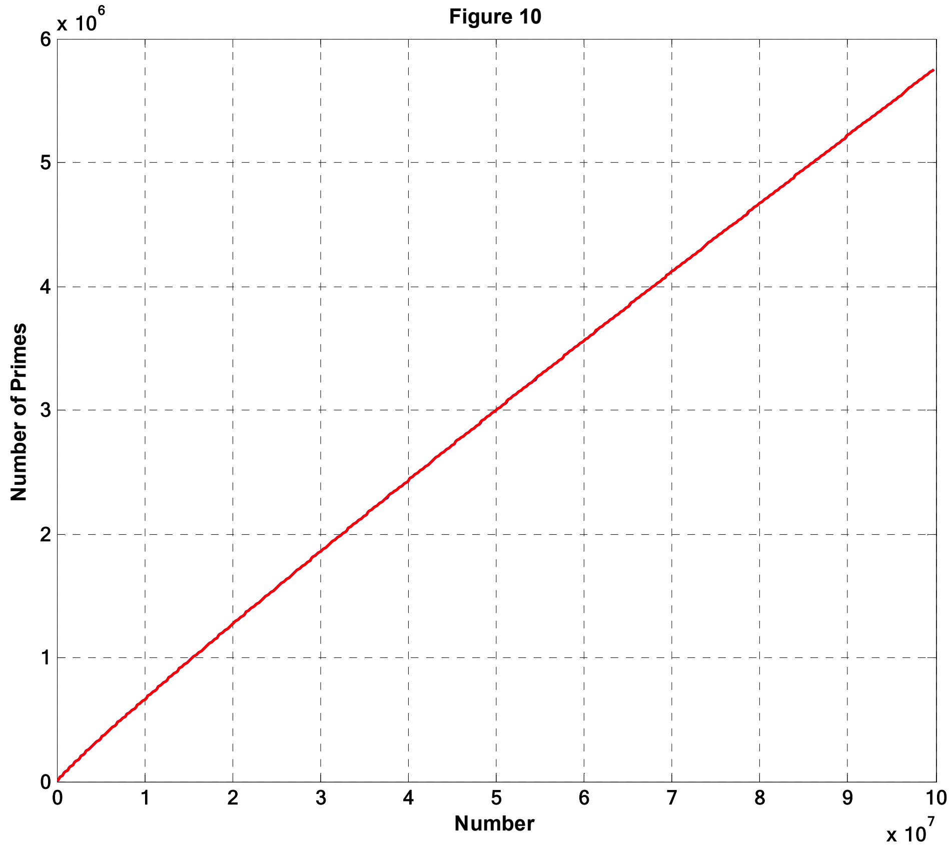
The value of b jitters as one would expect. It appears to have a slight downward trend. Eyeballing two points, $(0, 0.63)$ and $(10^8, 0.59)$, and taking a first order linear approximation yields

(9) $$b = 0.63 - 4 \times 10^{-10}x.$$

Putting Equation 9 into Equation 7 gives

(10) $$Lia(x) = \int_2^x \frac{1}{\ln(x)^{\left(1+\frac{1}{x^{(0.63-4\times10^{-10}x)}}\right)}} dt\ .$$

Figure 10 shows the computed value for the numbers of primes less than x, i.e. $Lia(x)$, to the actual values, i.e. $\pi(x)$. The range is $0\ to\ 100{,}000{,}000$ with a spacing of $200{,}000$.



The correlation is quite good. Admittedly, Equation 10 is ugly and nothing of great excitement was seen as would be in the mountain peaks of the math world. But the trek in the lower hills was fun and put to use the main hardware tool of the lowland trekker, the desk top computer. To go further would take the big boys with the big machines. A second order approximation of b would likely look like $b = c + \frac{d}{x}$ and would give better results at very large numbers. However, that ends the hike for today. It is late and time to return home as one must at the end of every trek.

Mathdrooler 25 Aug 2013