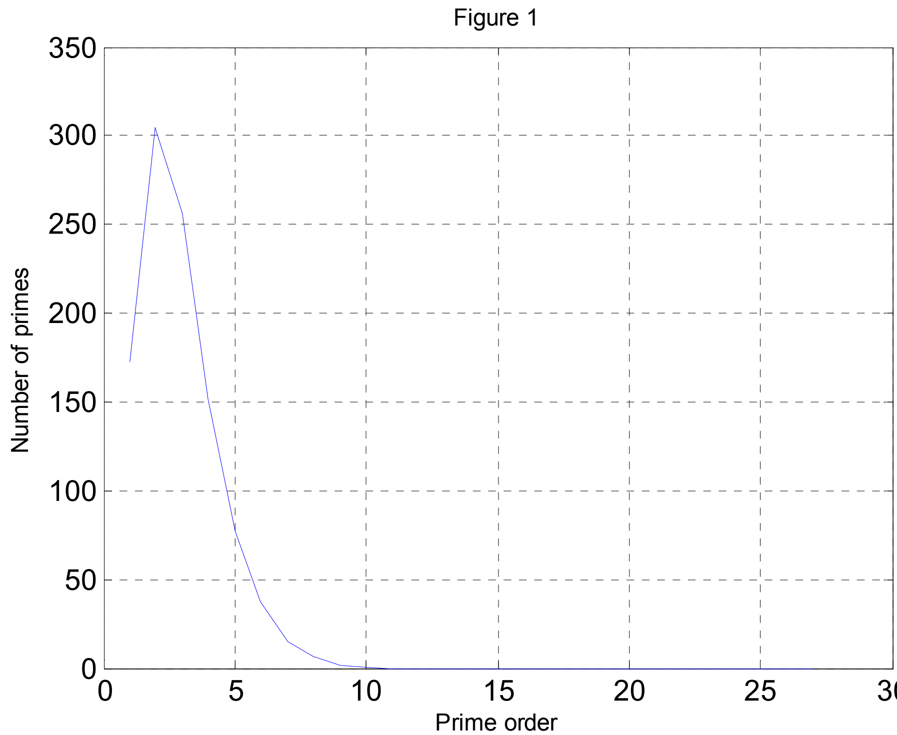
Prime Numbers

We are not going to shake the earth here. We are only going to explore the foot hills of the math landscape. There are many books on number theory of which the subject of primes will naturally be included. Except for those far below tree line they are laborious with few exciting views. We will hike and explore on the lower hills to see if there is anything new that is of interest.

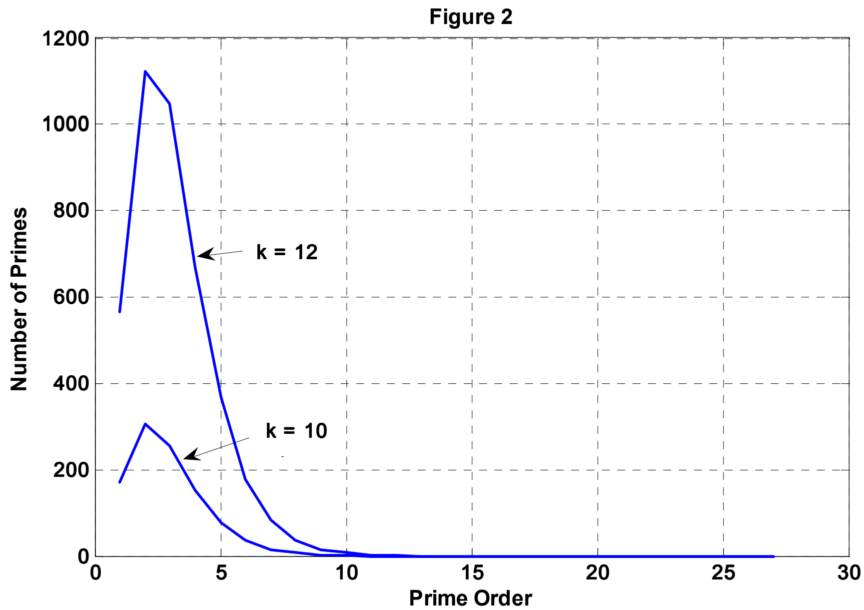
The subject of primes is akin to the subject of factoring. Before looking at either, the foundation must be clear. As said before the foundation of math is not set theory as taught in schools, but cows – or sheep or goats or some other kosher animal for those who are puritans. Math is rooted in the physical realm. The first question to arise is do primes have any significance in the physical world. In my limited knowledge of the physical world, I can think of none. Is there some place in the physical world that only a prime number of whatever will do. One could look into physics such as the number of particles in some entity, or in life science such as the number of pedals on a flower. Some flowers will have a prime number of pedals but not all. It is not a general rule necessary for flowers. The point is if prime numbers do not have any significance in the physical world and the foundation of math is the physical world then prime numbers are artificial.

A prime number is defined as a positive integer that can only be divided into integers by itself and 1. If divided by 1 the result is itself and if divided by itself the result is 1. From here on only integers and dividing into integers are considered. The definition of a prime will be slightly changed to reflect the foundation of math – that is cows. A rectangular array of objects is understood. It is objects arranged in a rectangle with width and length. A line of objects is also understood. The game will not be played that a rectangle with a width or length of 1 is a line. A rectangle has width and height greater than one. A line has only length. The same will be for a box of whatever dimension. If more rigor is wanted one must hike higher up on the math landscape. With all that, p is a prime number if p objects, cows for example, cannot be arranged into a rectangular array. Such a number will be called a prime of order 1. If p objects can be arranged into a rectangle but not a box, then p is a prime of order 2. In general if p objects can be arranged into an n-dimensional box but not an n plus 1 dimensional box then p is a prime of order n. This gives a physical meaning to the primeness of positive integers. In everyday math language every positive integer is a prime of order equal to the number of factors of the integer, the number 1 not being considered one of those factors. Let us take a look at the density of primes of various orders.

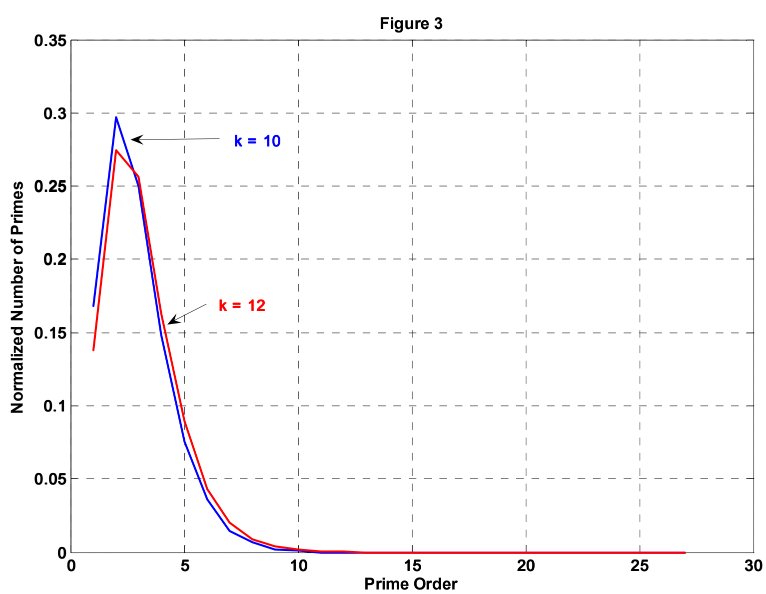
The first thing to do is to decide the range of numbers to look in. For example, one could determine the number of various orders of primes in the region from 2 to 100. However, it may be more productive to look in a range from 2 to 2^k. The highest prime order is k and there is only one prime of that order, which is 2^k. For $k=10$ the curve is shown in Figure 1.



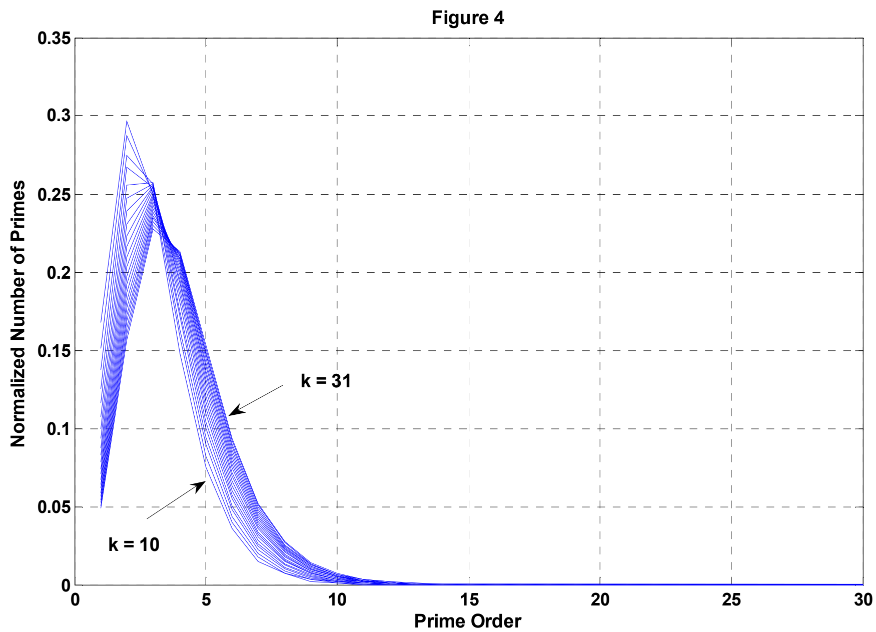
The above figure shows the distribution of prime orders up to 2^{10}. What does the distribution look like for higher numbers, say 2^{12}? Certainly, the curve will go further out. Figure 2 shows the comparison.



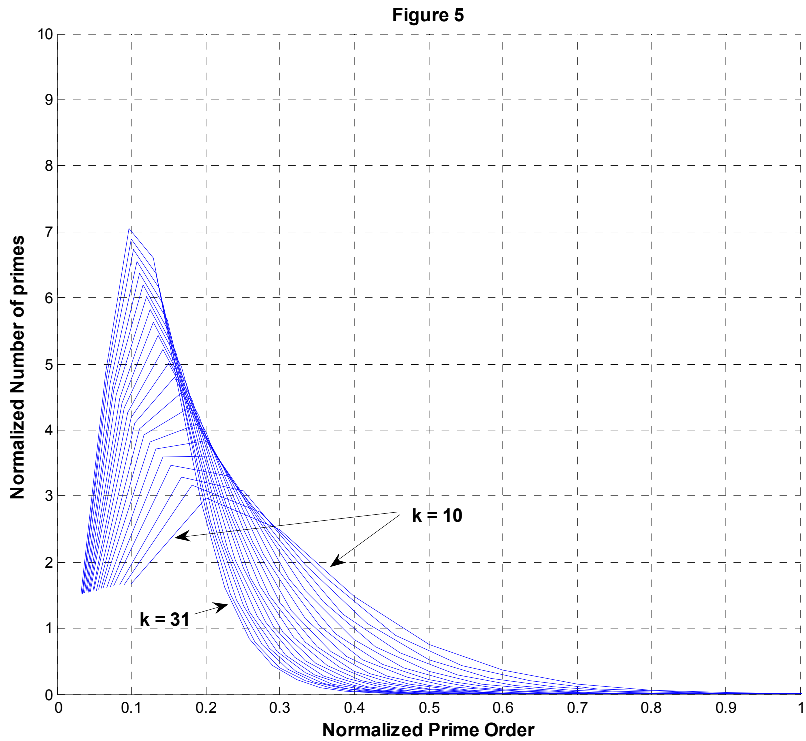
The shape of the curves look different, but it is hard to tell. For a better comparison the curves can be normalized so that the area under each curve is 1. To do so divide the number of primes by 2^{10} for the $k = 10$ curve and by 2^{12} for the $k = 12$ curve. Doing so gives Figure 3.



The curves look similar but are not the same. Look at a more extensive set of curves, say for k = 10 to 31. The curves are shown in Figure 4. (With Matlab one can get to k=27 in a reasonable time. Using Delphi 5 with class Tprimes by Gary Darby and Charles Doumar, one can get to k = 31.)



This does not look real exciting. On the x-axis each curve ends at k. To better compare the curves and find any trend, normalize the curves along the x-axis also making sure that the area under each curve is 1. Figure 5 shows what results.



One would like to define these curves by a function whose arguments are x and k. The curves raise and then fall. A curve that raises can have the form $y = x^m$. A falling curve can be $y = \frac{1}{n^x}$. A general function one may consider is $y = \frac{a(bx)^c}{c^{bx}}$. There are three unknowns a, b and c. Taking three points on a curve one should be able to solve for the unknowns for each curve. Let the three points on the curve be (x_1, y_1), (x_2, y_2) *and* (x_3, y_3). Without going into laborious detail one set of solutions is

$$a = y_1 \frac{c^{bx_1}}{(bx_1)^c}$$

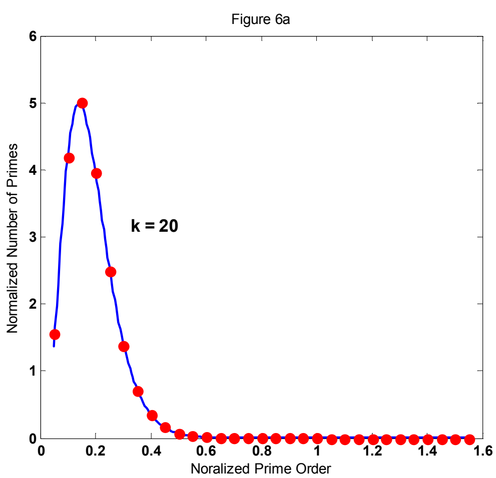
$$b = \frac{\log\left(\frac{y_2/x_1}{(x_2/x_1)^c}\right)}{(x_1 - x_2)\log(c)}$$

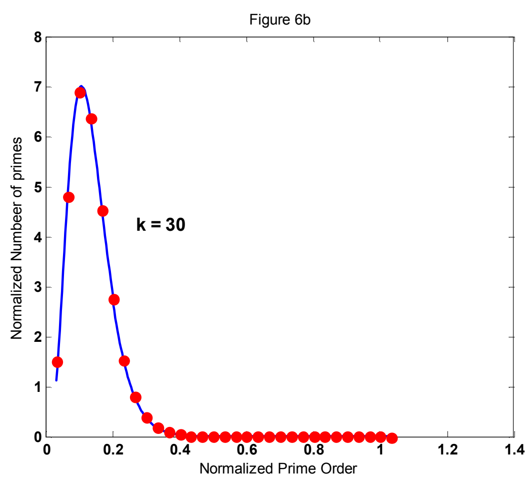
$$c = \left(\frac{y_3/y_1}{(x_3/x\_1)^c}\right)^{\frac{1}{b(x_1 - x_3)}}$$

One has a in terms of b and c, b in terms of c and c in terms of b and c. Having c one gets b. Having b and c one gets a. However c is in terms of b and itself. So an iterative process must be used. Continue the iteration until the values of d c do not change from five decimal places. Start with a value for c such that the iteration does not blow up, $c = 3$ is fine.

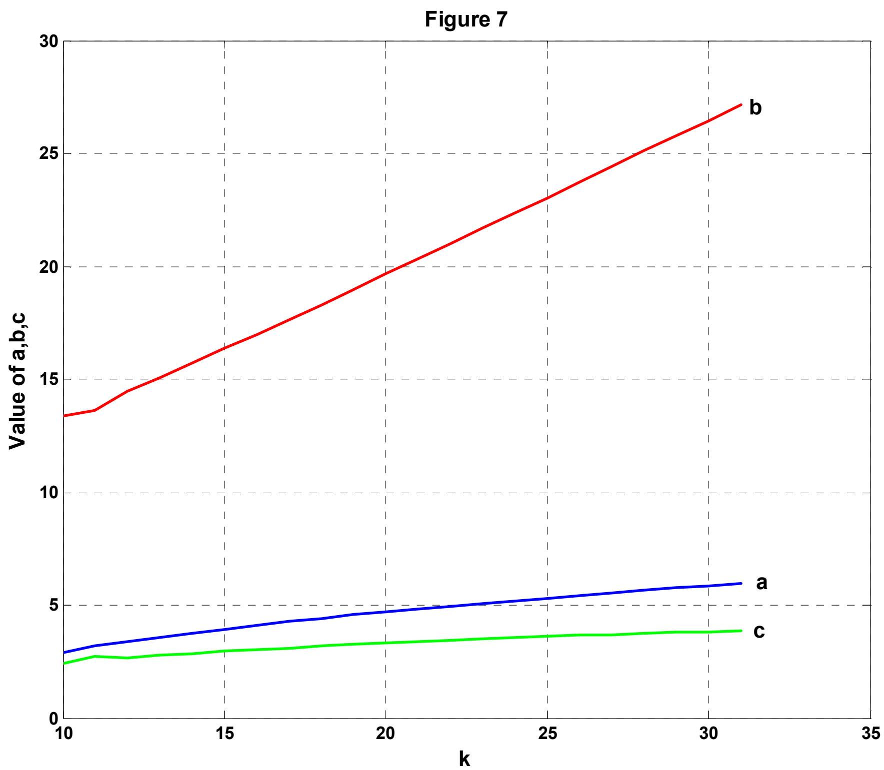
Before going on let's remember what one is looking at. The x-axis is the prime order. That is the number of factors, e.g. $12 = 2 \times 2 \times 3$ is of prime order 3. The y-axis is the number of numbers of a certain prime order. One or both axes may be normalized to give a better picture. Each curve covers the number range from 2 to 2 to 2^k .

The two plots below give a feeling of how well the curve estimates match the actual data. The red dots are the actual data and the blue curve the estimated values. As can be seen a fairly good match has been made. At least good enough for those who hike in the lower foot hills of the math world.

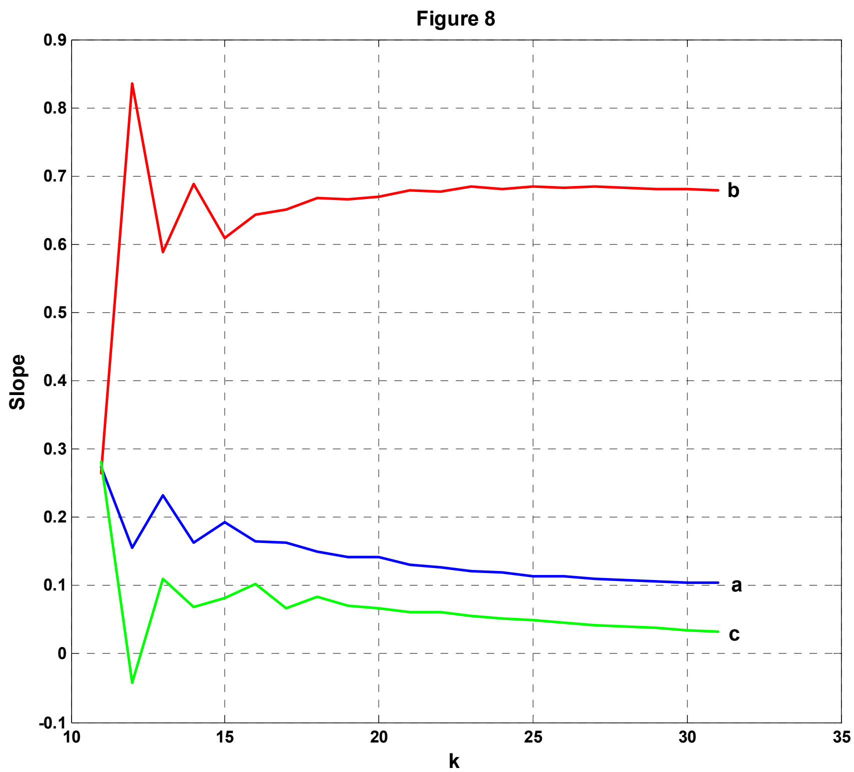




One would like to find the general trend for a, b and c. That is one would like to find a, b and c in terms of k. Here, one is endeavoring to do this experimentally. Unfortunately, even using Delphi 5 the highest k for which data could be obtained was $k = 31$. The figure below shows the values of a, b and c plotted against k.

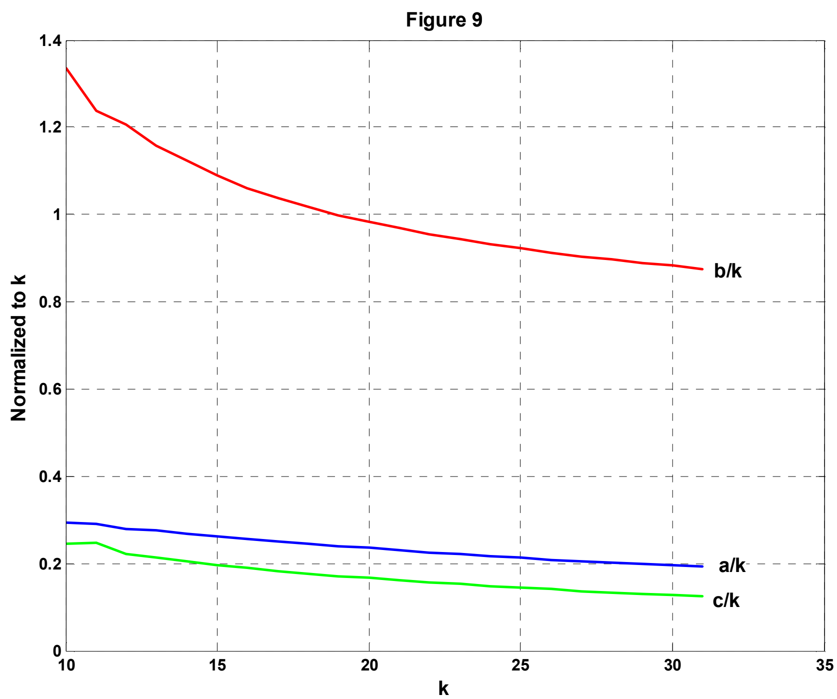


What can be said about the future for a, b and c ? The curve for b appears to be a straight line of increasing value. The curves for a and c appear to approach horizontal lines not to high above the value for $k = 31$. But looks can be deceiving. The next figure shows slopes for the three curves plotted against k .



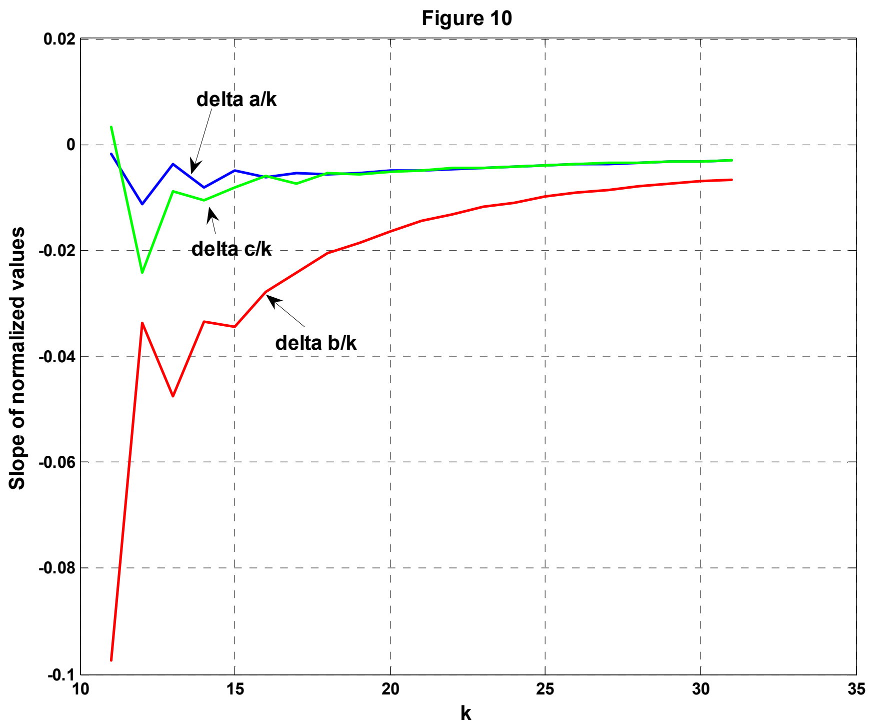
The b-slope curve could be horizontal. It also could be curving down slightly, meaning that the curve for b in Figure 7 would be curving down. The a-slope and c-slope curves appear to be parallel. A better look would help.

Normalize the curves by k. The result is shown in Figure 9.



The a/k and c/k curves appear to be parallel. They cannot go below zero and are unlikely to increase. They look as though (this is experimental math and one can use such language) they are approaching some horizontal lines. The b/k must also be approaching a horizontal line. That is they appear to be approaching asymptotes parallel to the k-axis. The slopes of the curves may give more insight.

Figure 10 shows the slopes of the curves in Figure 9.



From Figure 10 the slopes of a/k and c/k appear to be the same. The slope of b/k appears to parallel the slopes of a/k and c/k. If that is the case then from Figure 9

$$\frac{a}{k} - \frac{c}{k} = 0.0683 \qquad k = 31$$
$$c = a - 2.119$$

.

In Figure 8, the slopes of a, b and c appear to be parallel. If so,

$$\frac{\Delta b}{\Delta k} - \frac{\Delta a}{\Delta k} = 0.6786 - 0.1034$$
$$= 0.5752$$
$$b - a = 0.5752k + \gamma$$
$$b = a + 0.5752k + \gamma$$
$$b = a + 17.83 + \gamma \qquad where\ \ k = 31$$

Taking values of a and b for $k = 31$ from Figure 7,

$$27.143 = 5.936 + 17.83 + \gamma$$
$$\gamma = 3.349$$

If one knows a then b and c can be found. Not having any values for $k > 31$, and this being experimental math, an eyeball value from Figure 7 gives $a = 6$.

Remember the equation being considered for the normalized curve of the distribution of prime orders is

$$y = \frac{a(bx)^c}{c^{bx}}$$

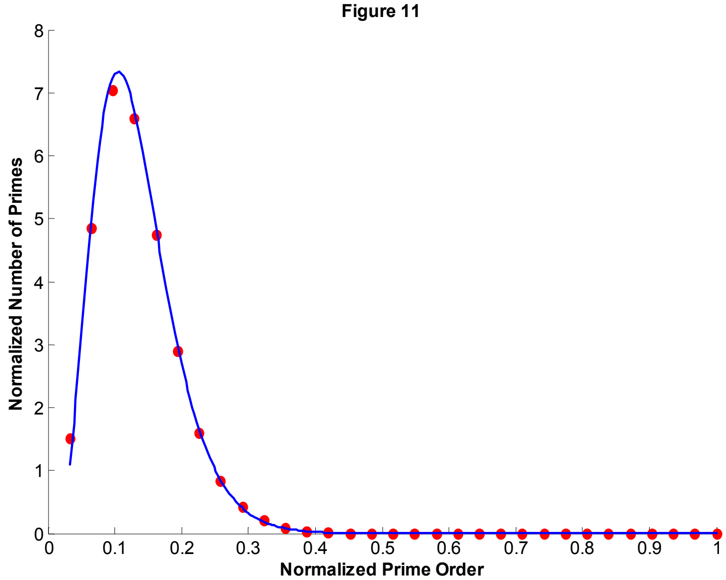
Putting in the values for b, c and γ gives

$$y = \frac{a\big((a + 60.78)x\big)^{a-2.119}}{(a - 2.119)^{(a+60.87)x}}$$

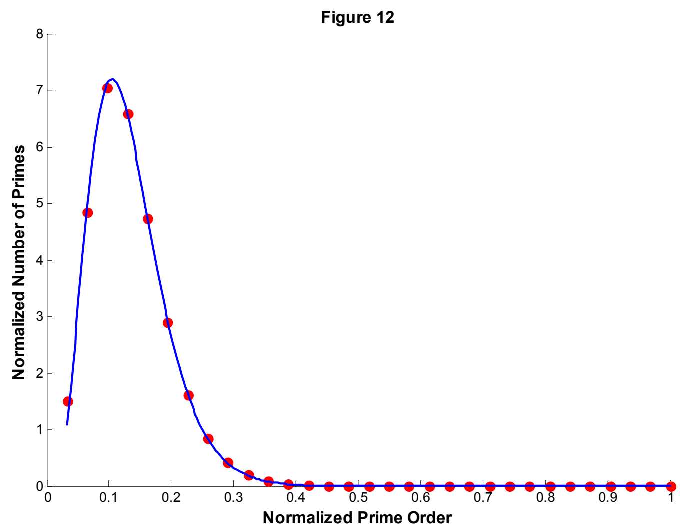
.

Taking the eyeball estimate $a = 6$ results in a pretty good estimate of the distribution of prime orders. Figure 11 shows the actual data as red dots and the estimated data as a blue curve for $k = 31$.

Figure 11



Taking $a = 5.962$ results in Figure 12, which is a better fit.



The highest value for which data could be obtained using my desk top machine was for $k = 31$. Rewrite a, b and c in terms of k.

$$a = 5.962$$

$$\gamma = 3.349$$

$$b = a + 0.5752k + \gamma$$

$$b = 9.311 + 0.5752k$$

$$c = a - 0.0683k$$

$$c = 5.962 - 0.0683k$$

And y becomes,

$$y = \frac{5.962\big((9.311 + 0.5752k)x\big)^{3.843}}{(3.843)^{(9.311+0.5752k)x}}$$

For $k = 31$, the above equation results in Figure 12 as it should. As stated before, $k = 31$ is the highest k obtained for my computing machine. If the eyeball assumptions are correct or fairly correct, the above equation can be used to extrapolate to higher values of k.

Figure 13 shows the prime order curve for $k = 100$.

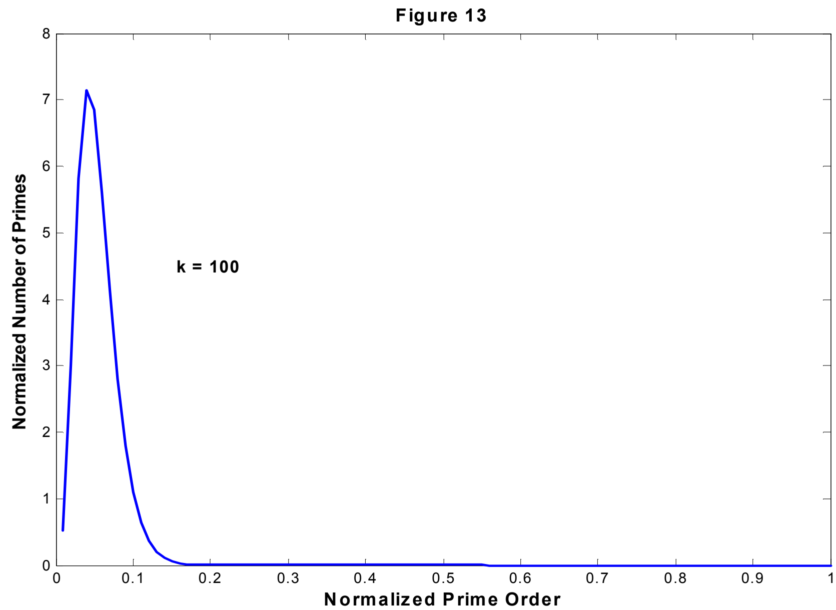
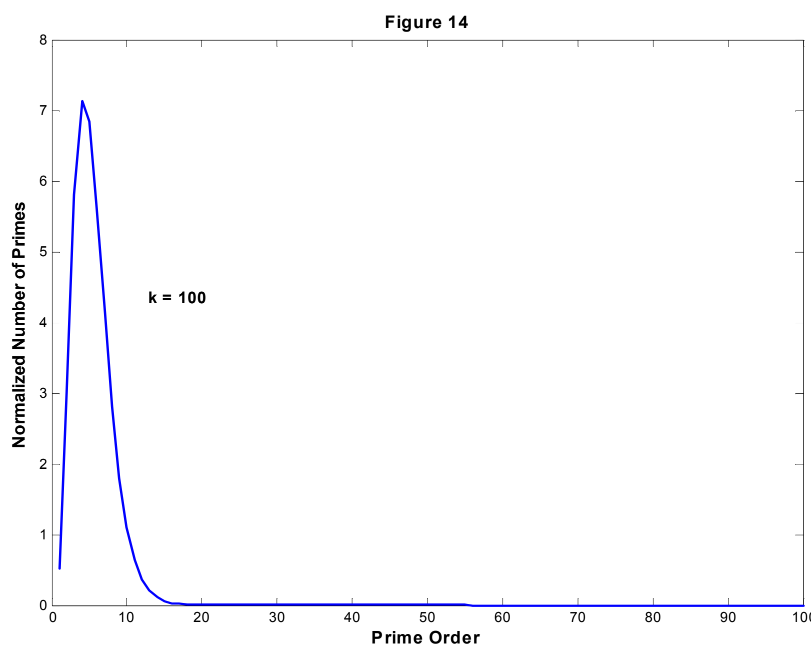


Figure 14 shows the same curve without the prime order being normalized.



The prime order with the greatest number of members for $k = 100$, that is up to 2^{100}, is prime order 4. Up to 2^{100} more numbers are composed of 4 factors than composed of any other number of factors. For $k = 1000$, i.e. up to 2^{1000}, prime order 5 has the greatest number of members as given by extrapolating the empirically derived prime order equation. The peak of the curve moves very slowly as k increases. A derivative can be taken of the prime order equation and set to zero to find the peak as a function of k. That will not be done here.

It should be recalled that the question at the beginning of the hike was if what are normally called prime numbers, prime numbers of order 1 here, are of artificial importance. Can prime numbers be found in the natural world, which is taken here as the foundation of math? Much has been written on prime numbers. Could the same to done for prime numbers of any order?

The sun is going down on the foothills of the math world and it is time to head back home. The hike was not as interesting as most, but some new landscape was seen. Returning to the prime number foothills will be more interesting.

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