Dynamic Series

To put the study of series on a firm foundation three propositions are needed.

1. *Little man cannot play God with the infinite*. The Creator will let little man see the infinite but will not let him touch it. This is important, attempted violations of it have led to some strange conclusions.

2. The equal sign "=" comes in two flavors, strong and weak. For the strong flavor, for any argument the value on the right side of the equal sign is equal to the value on the left side and vice versa. For the weak equal sign, there will be a range of arguments such that one side will produce a value and the other side will blow up to infinity. There will also be a range of arguments where both sides will be the same. The second necessary proposition is: *The weak equal sign does not exist in rigorous mathematics.*

3. *Ellipsis dots is a symbol of grammar, not of math.* While grammar has its element of rigor, math must be totally rigorous. Symbols of grammar can be incorporated into math only after they have been rigorously defined. Ellipsis dots have not been so defined. A rigorous definition of ellipsis dots is: Ellipsis dots when used in a series represent a finite number of terms of a series where the terms the ellipsis dots represent are clearly understood from the terms that precede the ellipsis dots and the term or terms that follow it.

With the propositions some examples can be looked at where unduly strange conclusions have been made. I will look at one here and then examine one that I take from the book "Prime Obsession".

A simple series many of us have seen in high school, at least in the high schools of forty years ago, is

$$\frac{1}{1+x} = 1 + x + x^2 + x^3 + \cdots$$

This violates all three propositions given above. The domain of the term on the left is $(x < 1, x > 1)$, i.e. (all x such that $x \neq 1$). The domain on the right is $(x < 1)$. This violates Proposition 2, which says that there is no weak equal sign. The ellipsis violates Proposition 3, which states it can only represent a finite number of terms and must be followed by at least one term. The ellipsis also violates Proposition 1 in that it somehow (in some vague sense) is infinity, which we cannot touch.

Using the three propositions the proper relation is

$$\frac{1}{1+x} = 1 + x + x^2 + x^3 + \cdots + \frac{x^k}{1-x} \quad for\ any\ k \geq 0$$

In this relation no claim is made that the ellipsis dots touch infinity. For $0 \leq x < 1$ the last term becomes vanishingly small as k gets larger. For $x > 1$ the sum of the terms on the right become increasingly larger as k gets larger. The last term becomes increasingly negatively large in such a way as to cancel all of the sum up to it, leaving $\frac{1}{1+x}$. All is well in the realm of math with no misbehavior.

The next example is from the book "Prime Obsession" and is what encouraged the investigation of the matter of series. The author presents an example of what mathematicians call a "conditionally congruent" series. It applies to series whose limit value depends on the order in which the terms are written. The example given is for the series of $\ln(2)$. It goes like so:

$$\begin{aligned}
\ln(2) &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots \quad which\ can\ be\ arranged \\
&= 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \cdots \\
&= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \cdots \\
&= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots \\
&= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right) \\
&= \frac{1}{2}\ln(2)
\end{aligned}$$

In light of the propositions this is immediately rejected. If true math is inconsistent and the sciences built on math are inconsistent also. Of course, this is just a mathematician's parlor trick, and not a very good one at that. The proliferation of twice as many minus signs as plus signs in the second and third equations is an obvious clue to the trick. Let's look at this with the three propositions in force.

Writing the series for $\ln(2)$, the first 10 terms gives

$$\ln(2)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+R_{10}$$

where R_{10} is the remainder of $\ln(2)$ after the n^{th} term. This will be rearranged into three lines. The first line will be what was devised above. The second line will be the terms not used in line 1 and not part of the remained. The third line will be the remainder. We then have

$$\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)=$$

(1a) $$\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\right)$$

(2a) $$\frac{1}{7}+\frac{1}{9}$$

(3a) $$R_{10}$$

Let's now add on more terms. We can add any number of terms we wish, but let's add four terms to keep things well-kept. Doing this results in

$$\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\left(\frac{1}{7}-\frac{1}{14}\right)=$$

(1b) $$\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}\right)$$

(2b) $$\frac{1}{9}+\frac{1}{11}+\frac{1}{13}$$

(3b) $$R_{14}$$

Adding another four terms yields

$$\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\left(\frac{1}{7}-\frac{1}{14}\right)-\frac{1}{16}+\left(\frac{1}{9}-\frac{1}{18}\right)=$$

(1c) $$\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}\right)$$

(2c) $$\frac{1}{11}+\frac{1}{13}+\frac{1}{15}+\frac{1}{17}$$

(3c) $$R_{18}$$

We see the pattern and can write the three lines more generally for the first $n = 2 + 4k$ terms where k is any positive integer.

(1d) $$\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\cdots+\frac{1}{\frac{n}{2}}\right)$$

(2d) $$\frac{1}{\frac{n}{2}+2}+\frac{1}{\frac{n}{2}+4}+\cdots+\frac{1}{n-1}$$

(3d) $$R_n$$

As n gets larger the first line approaches $\frac{1}{2}\ln(2)$ as we know. This is the mystical value it is claimed the rearrangement of the terms leads to. The third line can be looked at using our desk top number cruncher. As n gets larger, the sum of say the next one million terms after the n^{th} term appears to approach zero. Of course, these are not definite conclusions from mathematical theory, but are ever closer approximations from mathematical experiment using our desktop number smasher, somewhat analogous to a physicist getting ever closer approximations to some physical quantity using their atom smashers. Experiment can show the wonder of that which the theorist can abstract to boredom, at least in physics. Line two is the most interesting.

Line (2d) is a series somewhat different than any series I have encountered in my limited experience - it is dynamic. In most series the first term is constant and the n^{th} term can be written as a function of n. The value of the series, the sum of the terms, either approaches some finite number or infinity. The terms remain static while the sum changes as more terms are considered, that is as n increases. The limit to which the series in line two approaches is also investigated by increasing n, but here n has a different usage. It does not signify the n^{th} ther in the series. It signifies the range of the series and the value of each term with both the first and last terms changing as n^{th} changes. As n increases the first term in the series in line two becomes a higher term in the series for $\ln(2)$ and slips toward infinity out of touch. By Proposition 1 the Lord does let us see it though. Computing the sum of the series in line two for increasing n, one finds that the value of the series approaches $\frac{1}{2}\ln(2)$. It is the missing $\frac{1}{2}\ln(2)$ in the rearrangement of the terms in the series for $\ln(2)$. "Conditionally congruent" does not apply to this series and I suspect it does not apply to any series. It would seem that to change the value of a series of numbers by changing the order of the terms violates the laws of arithmetic. It is interesting that the series for $\frac{1}{2}\ln(2)$ can be static or dynamic. I suspect there is a rich world of dynamic series that has not been investigated.

Let's leave the question of "conditionally congruent" in relation to the series for $\ln(2)$ and simplify things a bit. In the dynamic series for $\frac{1}{2}\ln(2)$, replace $\frac{n}{2}$ with n. This can be done without losing generalization. It just means one is looking at a different series within the dynamic series. Instead of

$$\frac{1}{\frac{n}{2}+2}+\frac{1}{\frac{n}{2}+4}+\frac{1}{\frac{n}{2}+6}\dots+\frac{1}{n-1}$$

one has

$$\frac{1}{n+2}+\frac{1}{n+4}+\frac{1}{n+6}=\cdots=\frac{1}{2n-1}$$

.

Both of these series are within the dynamic series for $\frac{1}{2}\ln(2)$. (When I say dynamic series, I mean sequences of series, which in the present case are the series defined by the rules in line (2d) for $\frac{1}{2}\ln(2)$. With the knowledge gained, let's leave the above behind and look at a simpler, maybe the simplest, dynamic series.

Define a dynamic series $\widetilde{D}(\alpha{:}\beta{:}\gamma)$ by

(4) $$\widetilde{D}(\alpha{:}\beta{:}\gamma) = \frac{1}{\alpha} + \frac{1}{\alpha+\beta} + \frac{1}{\alpha+2\beta} + \cdots + \frac{1}{\alpha+\gamma\beta} \quad \mathit{where}$$

γ is a monotonically increasing function of α.

Call $\lim_{\alpha\to\infty} \widetilde{D}(\alpha{:}\beta{:}\gamma)$ the value of $\widetilde{D}(\alpha{:}\beta{:}\gamma)$ and denote it by $\overline{D}(\alpha{:}\beta{:}\gamma)$.

The dynamic series for $\frac{1}{2}\ln(2)$ can be written, somewhat awkwardly, as $\widetilde{D}\left(\frac{n}{2}+2{:}\,2{:}\,\frac{n-6}{4}\right)$. This puts a condition on n of $n = 4k+2$. The claim above is $\overline{D}\left(\frac{n}{2}+2{:}\,2{:}\,\frac{n-6}{4}\right) = \frac{1}{2}\ln(2)$.

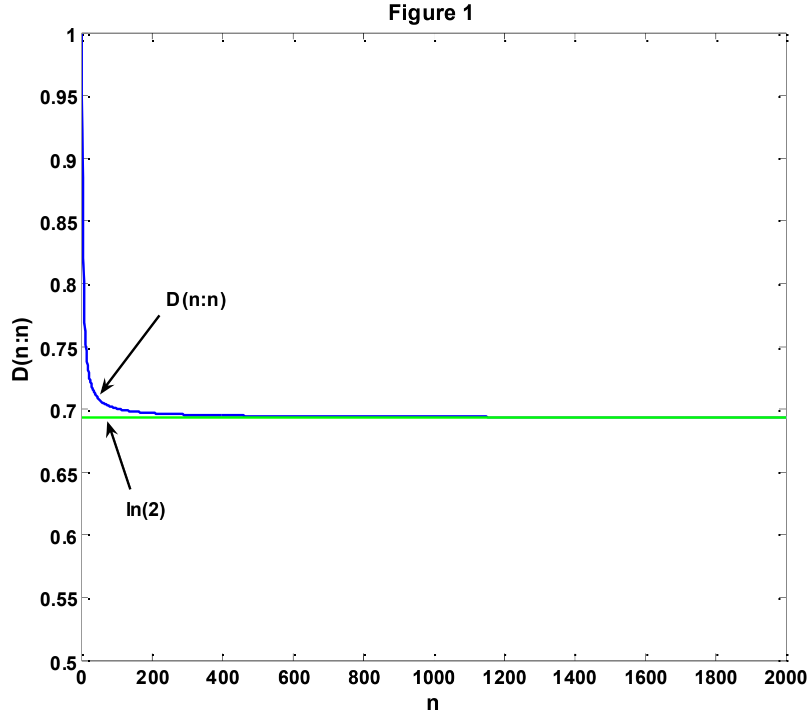
Having an idea of dynamic series, the task of investigating them can be made a little simpler by considering a simpler case. Consider the dynamic series $\widetilde{D}(n{:}\,1{:}\,n)$. From the definition above

(5) $$\widetilde{D}(n{:}\,1{:}\,n) = \frac{1}{n} + \frac{1}{1+n} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}.$$

Looking at ever larger n, one finds that $\widetilde{D}(n{:}\,1{:}\,n)$ approaches $\ln(2)$. Can it be proved that $\overline{D}(n{:}\,1{:}\,n) = \ln(2)$? Such a proof has not been looked at and may be difficult. It is suspected that $\overline{D}(n{:}\,1{:}\,n) = \ln(2)$. (From here on if $\beta = 1$, write $\overline{D}(n{:}n)$ with the understanding that $\overline{D}(n{:}\,n) = \overline{D}(n{:}\,1{:}\,n)$.)

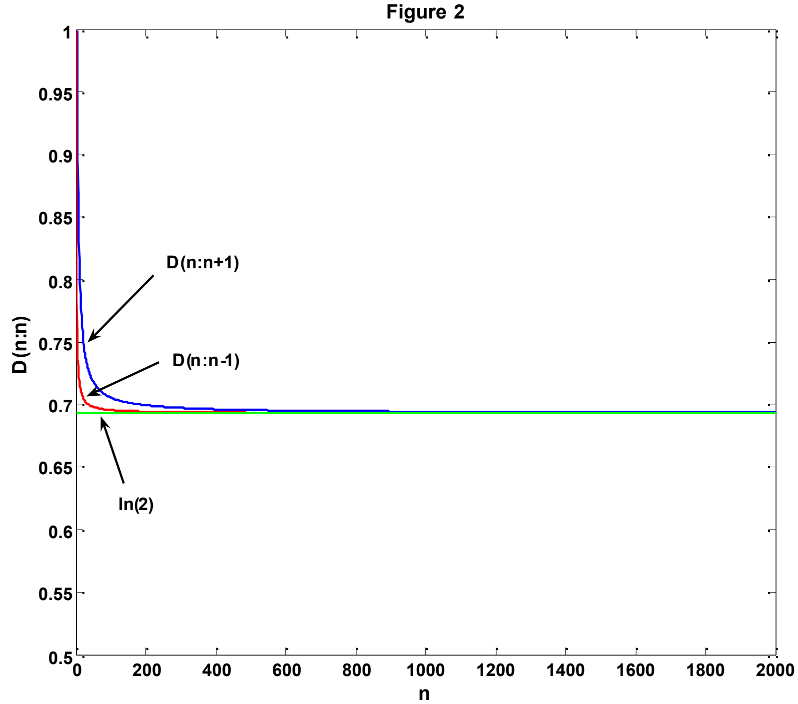
What one would first like to know is if $\overline{D}(\alpha{:}\,\beta{:}\gamma)$ is finite. It is not as easy to show this as a simple ratio test for static series, but is not too much more difficult for a certain simple dynamic series such as $\widetilde{D}(n{:}\,n)$. We know from construction above that $\overline{D}\left(\frac{n}{2}+2{:}\,2{:}\,\frac{n-6}{4}\right)$ is finite and likely has a value of $\frac{1}{2}\ln(2)$. This was not directly proved. Leaving the question of proof, which math droolers can do, to others, turn instead to the desktop number cruncher. The plot below shows $\widetilde{D}(n{:}\,n)$ plotted against n and compared to $\ln(2)$.

Figure 1



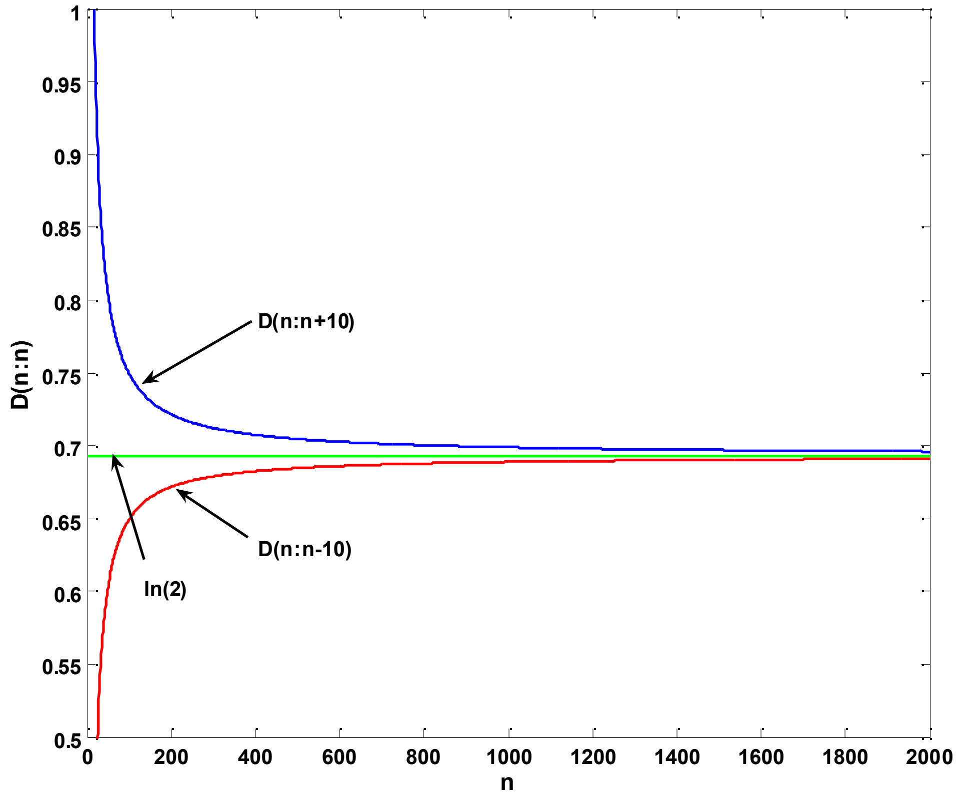
What if a term is added to or taken away from the series? Below are plots of $\widetilde{D}(n{:}\,n+1)$ and $\widetilde{D}(n{:}\,n-1)$. (It will loosely be said that $\widetilde{D}$ is being plotted, the meaning being obvious.)

Figure 2



Now look at ten terms added or taken away. Below are plots of $\widetilde{D}(n:n+10)$ and $\widetilde{D}(n:n-10)$.

Figure 3

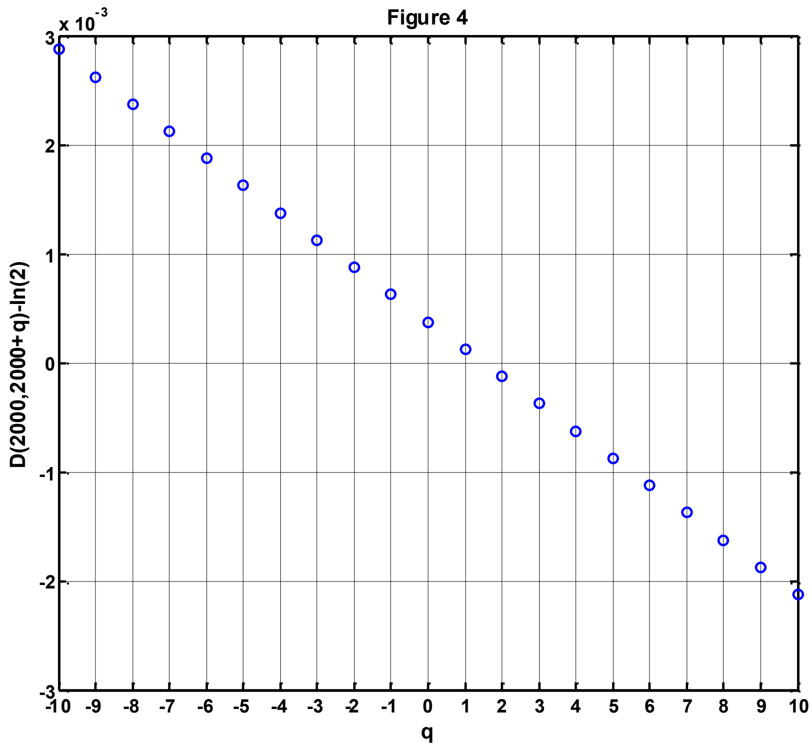


Adding or taking away terms does not seem to change the value of the dynamic series. In fact

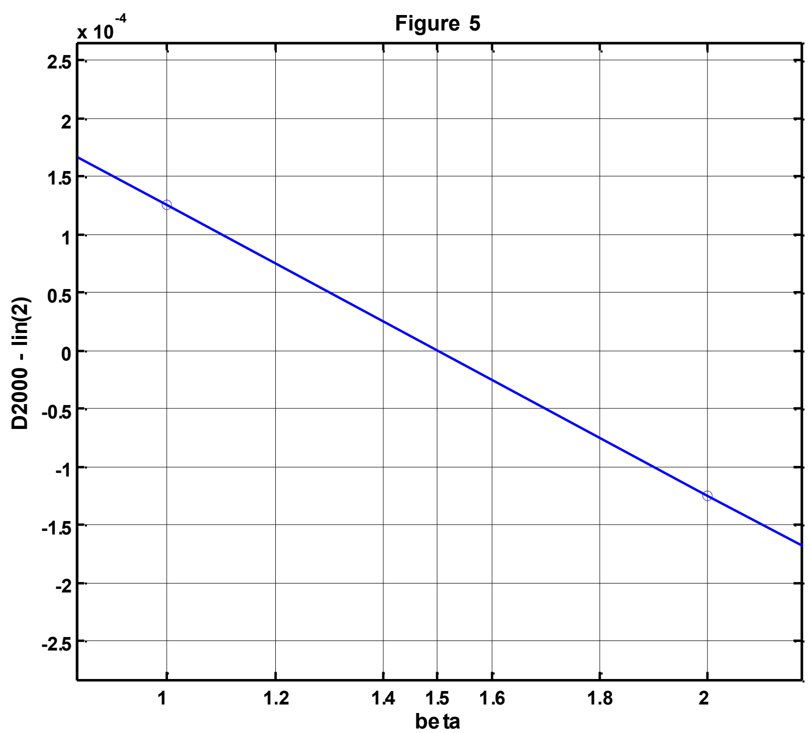
$$\bar{D}(n:n) = \bar{D}(n+p:n+q) \;\; where\ q > p$$

A formal proof is not needed. Clearly, the difference between the dynamic series in the plot above is a finite number of terms, each of which approaches zero as n gets larger.

In Figure 3 it is noticed that the dynamic series approach $\ln(2)$ from different directions. $\widetilde{D}(n:n+10)$ approaches from above and $\widetilde{D}(n:n+10)$ approaches from below. Where does the approach change? That is, for what q in $\widetilde{D}(n:n+q)$ does the approach change from above to below? Figure 4 shows $\widetilde{D}(2000:2000+q) - \ln(2)$ for $-10 \leq q \leq 10$.



The crossing point is seen to be between $q = 1$ and $q = 2$. Expanding Figure 4 and drawing a line through the points gives Figure 5 below.



The line appears to cross the zero point at $q = 1.5$. If the figure is further enlarged, it will be found that the crossing point is just left of $q = 1.5$. However, taking larger n puts the crossing point ever closer to $q = 1.5$. The point at which the line crosses at $\bar{D}(n{:}\, n+q) - ln(2)$ appears to approach $q = 1.5$ as $n \to \infty$. For finite n the crossing point is near $q = 1.5$. Remember, the crossing point is the q for which $\bar{D}(n{:}\, n+q)$. But, $n + q + 1$ is the number of terms in $\tilde{D}(n{:}\, n+q)$. A term is either there or not there. What does half, or a fraction, of a term mean? That fraction of a term is irrational since $\ln(2)$ is irrational and $\bar{D}(n{:}\, n+q)$ is rational for any n. The half term's value for any n is $\bar{D}(n{:}\, n+q) - \ln(2)$. The crossing point is just a curiosity off the trail we are hiking and will be left as that.

Before leaving our day hike in the foothills of the math world consider $\tilde{D}(n{:}\, 2n)$. What one will find is that $\bar{D}(n{:}\,1{:}\,2n) = \ln(3)$. In general one will find that $\bar{D}(n{:}\,1{:}\,k+n) = \ln(k+1)$ where k is a positive integer. To be more bold, $\bar{D}(n{:}\,1{:}\,round(x+n)) = \ln(x+1)$ where $x > 1$ i.e. x is any positive real number greater than 1. The third term is rounded because the number of terms is an integer. We still do not know what a fraction of a term means. One can also find, at least experimentally, that $-\bar{D}(xn{:}\,1{:}\,n - xn) = \ln(x)\ where\ 0 < x < 1$.

Define the dynamic series for the ln function in a way that is slightly removed from its physical founding but easier to work with:

$$\tilde{L}(\alpha{:}\,\beta) = \frac{1}{\alpha} + \frac{1}{\alpha+1} + \frac{1}{\alpha+2} + \cdots + \frac{1}{\beta-1} + \frac{1}{\beta}\quad where\ \alpha < \beta$$

Then $\bar{L}(n{:}\,xn) = \ln(x)\ where\ x \geq 1$ and $-\bar{L}(xn{:}\,n) = \ln(x)\ where\ 0 < x < 1$.

Recall that a dynamic series is a sequence of series related to an integer n such that as n gets larger the first and last terms in the series move to the right along the number scale. We lose touch of the series but we can see as its value approaches a finite number. The sequence of series examined was such that to get to the next member series from the current one a term was removed from the front and terms were added to the end. See below for two consecutive members of the dynamic series $\tilde{L}(n{:}\,2n)$.

$$n = 3 \qquad \tilde{L}(3{:}\,6) = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$$
$$n = 4 \qquad \tilde{L}(4{:}\,8) = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

This type of dynamic series will be called single track. The table below shows the thinking behind the name. The $x's$ show the positions along the track for three consecutive series of $\tilde{L}(n{:}\,2n)$. The series moves away and spreads out similar to a wave packet moving in time.

Track		1	1/2	1/3	1/4	1/5	1/6	1/7	1/8	1/9	1/10	1/11	...
n=3				x	x	x	x						
n=4					x	x	x	x	x				
n=5						x	x	x	x	x	x		

For a single track dynamic series a test can be made to determine if its value is finite. As $n \to \infty$ if the ratio of the added terms to the removed term as n increases to $n+1$ is 1, the dynamic series approaches a finite value. For $\tilde{L}(n, 2n)$ that ratio is

$$\lim_{n\to\infty} \frac{\dfrac{1}{2(n+1)-1} + \dfrac{1}{2(n+1)}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{4n^2 + 3n}{4n^2 + 6n + 2} = 1$$

An example of a multi track dynamic series is

$$\tilde{A} = \frac{n}{n^2} + \frac{n+1}{n^2+1} + \frac{n+2}{n^2+2} + \cdots + \frac{n+n}{n^2+n}$$

All the terms will vary from one series to the next as n increases while $\tilde{A}(n) = 1.5$

There are likely many interesting dynamic series to be discovered. But the sun will be going down soon and I must get back to the trail head and return to home after a nice hike in the lower trails on the landscape of the math world.

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