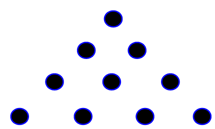
Triangle Numbers

A triangle number $n, where\ n \geq 0$ is defined by

(1) $$\boxed{n^{\Delta} = 1 + 2 + 3 + \cdots + n.}$$

One can see why such numbers are called triangle by physically looking at one such number, say 4^{Δ} .



As a new row is added the number of dots in the row is increased by 1. The number of dots along each side is the same.

It can be easily shown that

(2) $$\boxed{n^{\Delta} = \frac{n(n+1)}{2}.}$$.

This can be naturally extended to $(-n)^{\Delta}$, where $n \geq 0$, by

(3) $$\boxed{(-n)^{\Delta} = \frac{(n-1)n}{2}.}$$.

The illustration above is a physical example of a triangle number. What is the physical example of a negative triangle number? Because the concept of a negative number is so familiar, it is not thought of as being abstract. Negative numbers was a difficult concept to accept when it first appeared. What is less than nothing? A square number can be physically looked at as a square array of say dots. What is the physical appearance of a negative number squared? The same applies to triangle numbers. We have gone from the physical to the abstract. One can easily slip from math to philosophy. In that philosophy is the playground of those who don't know but want to sound like they do and nothing ever comes of it, on to something concrete.

Now that we have the definitions down, let's look at some relationships. We can start with an easy one. Consider $(cd)^{\Delta}$ and proceed as follows:

$$
\begin{aligned}
(cd)^{\Delta} &= \frac{(cd)(cd+1)}{2} \\
&= \frac{cd\big((c+1)(d+1)-(c+1)(d+1)+cd+1\big)}{2} \\
&= \frac{cd\big((c+1)(d+1)-cd-c-d+1\big)}{2} \\
&= \frac{cd\left[\frac{(c+1)(d+1)}{2}-\frac{(c+1)(d+1)}{2}+cd+1\right]}{2} \\
&= \frac{c(c+1)}{2}\frac{(d+1)}{2}+cd\left(-\frac{cd+c+d+1}{4}+\frac{cd+1}{2}\right) \\
&= c^{\Delta}d^{\Delta}+cd\left(\frac{-cd-c-d-1+2cd+2}{4}\right) \\
&= c^{\Delta}d^{\Delta}+cd\left(\frac{cd-c-d+1}{4}\right) \\
&= c^{\Delta}d^{\Delta}+cd\frac{(c-1)(d-1)}{4} \\
&= c^{\Delta}d^{\Delta}+cd\frac{(c-1)c}{2}\frac{(d-1)d}{2} \\
&= c^{\Delta}d^{\Delta}+(c-1)^{\Delta}(d-1)^{\Delta}
\end{aligned}
$$

(4) $$\boxed{(cd)^{\Delta}=c^{\Delta}d^{\Delta}+(c-1)^{\Delta}(d-1)^{\Delta}}$$

This can be compared with the similar relationship for squared numbers:

$$(cd)^2 = c^2d^2 \ .$$

The notation with the triangle symbol being in the same place the power symbol resides was done owing to the similarity of the relationships.

Take $c \geq d$ and manipulate the terms as follows:

$$
\begin{aligned}
(cd)^{\Delta} &= c^{\Delta}d^{\Delta} + (c-1)^{\Delta}(d-1)^{\Delta} \ \ where\ c \geq d \\
c^{\Delta}d^{\Delta} &= (cd)^{\Delta} - (c-1)^{\Delta}(d-1)^{\Delta} \\
&= (cd)^{\Delta} - [(c-1)(d-1)]^{\Delta} + (c-2)^{\Delta}(d-2)^{\Delta} \\
&= (cd)^{\Delta} - [(c-1)(d-1)]^{\Delta} + [(c-2)(d-2)]^{\Delta} - (c-3)^{\Delta}(d-3)^{\Delta} \\
&\vdots \\
&= (cd)^{\Delta} - [(c-1)(d-1)]^{\Delta} + [(c-2)(d-2)]^{\Delta} \\
&\qquad - \cdots (-1)^{d-2}\big[\big(c-(d-2)\big)\big(d-(d-2)\big)\big]^{\Delta} \\
&\qquad + \ (-1)^{d-1}((c-(d-1))^{\Delta}\big(d-(d-1)\big)^{\Delta} \\
&= (cd)^{\Delta} - [(c-1)(d-1)]^{\Delta} + [(c-2)(d-2)]^{\Delta} \\
&\qquad - \cdots (-1)^{d-2}[(c-d+2)(2)]^{\Delta} + \ (-1)^{d-1}((c-d+1)^{\Delta}
\end{aligned}
$$

(5)

$$
\boxed{
\begin{aligned}
c^{\Delta}d^{\Delta} = (cd)^{\Delta} &- [(c-1)(d-1)]^{\Delta} + [(c-2)(d-2)]^{\Delta} \\
&- \cdots (-1)^{d-2}[(c-d+2)(2)]^{\Delta} + \ (-1)^{d-1}((c-d+1)^{\Delta}
\end{aligned}
}
$$

What we find is that the produce of two triangle numbers can be expressed as a series of triangle numbers. Not earth shaking, but interesting. Maybe something a little more interesting can be found.

For square integer numbers $a = b^2$ one gets $b = a^{1/2}$ that is either an integer or irrational number. Is there a similar relation for triangle numbers? That is, if $c = d^{\Delta}$ what does $d = c^{1/\Delta}$ look like? To find out do the following:

$$c = d^{\Delta}$$

$$= \frac{d(d+1)}{2}$$

$$d^2 + d - 2c = 0$$

$$d = \frac{-1 \pm \left(1 - 4(-2c)\right)^{1/2}}{2}$$

(6) $$\boxed{c^{1/\Delta} = \frac{-1 \pm (1+8c)^{1/2}}{2}}$$

This can be further manipulated:

$$2c^{1/\Delta} = -1 \pm (1 + 8c)^{1/2}$$

$$(1 + 8c)^{1/2} = \pm(2c^{1/\Delta} + 1)$$

Let $b = 1 + 8c$ then

(7) $$\boxed{b^{1/2} = \left[2\left(\frac{b-1}{8}\right)^{1/\Delta} + 1\right]}$$

An observation can be made from equation (6). If c is a triangle number, then $c^{1/\Delta}$ is an integer and it follows that $1 + 8c$ is a square odd number. It can also be shown that if $1 + 8c$ is a square odd number than c is a triangle number. Or, if b is a square odd number then $\frac{b-1}{8}$ is a triangle number.

It should be noted that triangle numbers were constructed from integers. However, in the equations above one can put in any real number. A triangle number was defined as the sum of a series and it was found to be equal to $\frac{c(c+1)}{2}$. Any number for c will yield a value for the term, extending the concept of Δ. What follows can be applied to all real numbers.

Consider $(a+b)^{\Delta}$ where $a > b$.

$$\begin{aligned}(a+b)^{\Delta} &= 1+2+3+\cdots+b+\cdots+a+(a+1)+\cdots+(a+b)\\ &= a^{\Delta}+(a+1)+(a+2)+\cdots+(a+b)\\ &= a^{\Delta}+(a+a+\cdots a)+(1+2+\cdots+b)\quad b\ munber\ of\ a^{'}\sin second\ term\\ &= a^{\Delta}+ab+b^{\Delta}\end{aligned}$$

(8) $$\boxed{(a+b)^{\Delta} = a^{\Delta}+ab+b^{\Delta}}$$

This can be compared to the similar relation for squaring the sum of two numbers.

$$(a+b)^2 = a^2+2ab+b\text{^}2$$

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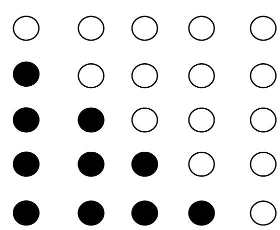
For the Math Drooler, discovering relationships or properties physically or by construction is more interesting than formal abstract proof. And it may be more instructive, giving a physical feeling behind the math. A good example of a physical proof is the following.

Consider a square composed of 25 dots.



The dots are shaded to show two triangles. One triangle has a side of 5 and the other has a side of 1 less than 5. So we see that $5^2 = 5^{\Delta}+4^{\Delta}$. In general

(9) $$n^2 = n^{\Delta}+(n-1)^{\Delta}$$

And that is all the proof needed, although the same could be easily shown from the formula for a triangle number and would more satisfy the discriminating mathematician.

Now, consider $n = ab$ where $b > 2a$ and a, b are odd integers. Let $k = b - 2a$. Then $k > 0$ and $b = 2a + k$. The following may be shown.

$$
\begin{aligned}
n &= ab \\
&= a(2a + k) \\
&= a(2a + 1 + k - 1) \\
&= \frac{(2a)\big((2a) + 1\big)}{2} + a(k - 1) \\
&= (2a)^{\Delta} + (2a)\frac{k - 1}{2} \\
&= (1 + 2 + \cdots + 2a) + \left(\frac{k - 1}{2} + \frac{k - 1}{2} = \cdots = \frac{k - 1}{2}\right) \ \{2a\ terms\} \\
&= \left(1 + \frac{k - 1}{2}\right) + \left(2 + \frac{k - 1}{2}\right) + \cdots + \left(2a + \frac{k - 1}{2}\right) \\
&= \left(2a + \frac{k - 1}{2}\right)^{\Delta} - \left(1 + \frac{k - 1}{2} - 1\right)^{\Delta} \\
&= \left(2a + \frac{b - 2a - 1}{2}\right)^{\Delta} - \left(\frac{b - 2a - 1}{2}\right)^{\Delta} \quad since\ k = b - 2a \\
&= \left(\frac{b - 1}{2} + a\right)^{\Delta} - \left(\frac{b - 1}{2} - a\right)^{\Delta}
\end{aligned}
$$

(10) $$\boxed{n = \left(\frac{b-1}{2} + a\right)^{\Delta} - \left(\frac{b-1}{2} - a\right)^{\Delta}}$$

Consider $n = ab$ where $b < 2a$ and a, b are odd integers. Let $k = 2a - b$. Then $k > 0$ and $b = 2a - k$. Proceeding as was done above:

$$\begin{aligned}
n &= ab \\
&= a(2a - k) \\
&= a(2a + 1 - k - 1) \\
&= \frac{(2a)(2a+1)}{2} + a(k+1) \\
&= (2a)^{\Delta} + (2a)\frac{k+1}{2} \\
&= (1 + 2 + \cdots + 2a) - \left(\frac{k-1}{2} + \frac{k-1}{2} + \cdots + \frac{k-1}{2}\right) \quad \{2a\ terms\} \\
&= \left(1 - \frac{k+1}{2}\right) + \left(2 - \frac{k+1}{2}\right) + \cdots + \left(2a - \frac{k+1}{2}\right) \quad partial\ series \\
&= \left(2a - \frac{k+1}{2}\right)^{\Delta} - \left(1 - \frac{k+1}{2} - 1\right)^{\Delta} \\
&= \left(2a - \frac{2a-b+1}{2}\right)^{\Delta} - \left(-\frac{2a-b+1}{2}\right)^{\Delta} \quad since\ k = 2a - b \\
&= \left(\frac{b+2a-1}{2}\right)^{\Delta} - \left(\frac{-2a+b-1}{2}\right)^{\Delta} \\
&= \left(\frac{b-1}{2} + a\right)^{\Delta} - \left(\frac{b-1}{2} - a\right)^{\Delta} \\
&= \left(\frac{b-1}{2} + a\right)^{\Delta} - \left((a-1) - \frac{b-1}{2}\right)^{\Delta} \quad where\ (-n)^{\Delta} = (n-1)^{\Delta}
\end{aligned}$$

(10) $\boxed{n = \left(\frac{b-1}{2} + a\right)^{\Delta} - \left(\frac{b-1}{2} - a\right)^{\Delta}}$ $where\ b < 2a\ and\ a, b\ are\ odd\ integers$

(10b) $\boxed{n = \left(\frac{b-1}{2} + a\right)^{\Delta} - \left((a-1) - \frac{b-1}{2}\right)^{\Delta}}$

Any number that can be factored into two odd integers is the difference of two triangle numbers. This same relation, which is easier to prove, holds for the difference of square numbers. That is, any number that can be factored into two odd integers is the difference of two square numbers.

Triangle numbers may be a little more interesting in that there appears to be more than one combination the factors can be dissolved into:

(11) $$\boxed{\begin{aligned} n &= ab \\ &= \left(b+\frac{a-1}{2}\right)^{\Delta} - \left(b-\frac{a+1}{2}\right)^{\Delta} \end{aligned}}$$

.

I will leave (11) to the reader to prove.

Look at Equations 10 and 11 closer as they are rewritten below.

(10) $$n = \left(\frac{b-1+2a}{2}\right)^{\Delta} - \left(\frac{b-1-2a}{2}\right)^{\Delta}$$

(11) $$n = \left(\frac{a-1+2b}{2}\right)^{\Delta} - \left(\frac{-a-1+2b}{2}\right)^{\Delta}$$

In the derivations above the number n was composed of two odd factors a and b. Although I did not state it, b was considered to be the greater of the factors and either greater than or less than $2a$. Equations 10 and 11 give two distinct results for $n = c^{\Delta} - d^{\Delta}$. Symmetry is esthetically pleasing in mathematics and in the physical world. What would be esthetically satisfying would be to replace a and b in Equation 10 and obtain Equation 11. That is not the case. In the first term on the right of each equation a can be replaced with b and b with a. The first term is symmetric with a and b. The second term in each equation is not as cooperative. In that term a is replaced with $-b$ and b is replaced with $-a$. This is a symmetry, but not as pleasing as in the first term. The first and second terms have different symmetries. This is not something expected in simple math as enjoyed here, but in the less friendly world of physics.

Before moving on a few results can be listed. Rewrite Equations 10 and 11 as

$$\begin{aligned} &n = ab \\ &n = c_1^{\Delta} - d_1^{\Delta} \;\; from\; 10 \\ &n = c_2^{\Delta} - d_2^{\Delta} \;\; from\; 11 \end{aligned}$$

The following relations can be stated:

$$\begin{aligned} c_1 &= \frac{b-1}{a} + a & \qquad d_1 &= \frac{b-1}{2} - a \\ c_2 &= b + \frac{a-1}{2} & \qquad d_2 &= b - \frac{a+1}{2} \\ c_2 &= \frac{5c_1 + 3d_1 + 2}{4} & \qquad d_2 &= \frac{3c_1 + 5d_1 + 2}{4} \end{aligned}$$

c_1 and d_1 are constrained by the factors of n, a and b, which are taken to be odd numbers. From either relation

$$c_1 - d_2 = 2a$$
$$c_1 + d_1 = b - 1$$

the constraint on c_1 and d_1 is that they are either both even integers or both odd integers.

Define a triangle offset number to be $c^\Delta - d^\Delta$. Use the notation $\Delta_d^c = c^\Delta - d^\Delta$.

It can be written

(12) $$n = ab = \Delta_d^c \quad \forall \ odd\ a, b$$

and shown that $$\Delta_d^c = \frac{(c-d)(c+d+1)}{2}$$

.

Equation 12 says that any number that can be factored into two odd integers is an offset triangle number. The number 1 being one of the factors, such as in $n = 7 = 7 \times 1$, is the trivial case in that every number is a triangle number of offset 1.

A triangle number n is the sum of all integers from 1 to n. Instead of summing all the numbers, what if n numbers are summed from 1 to n spaced by 2. What if the spacing is 3 or say some β. The notation and definition will be

(13) $$n^{\Delta_\beta} = 1 + (1+\beta) + (1+2\beta) + (1+3\beta) + \cdots + (1+(n-1)\beta)$$

If $\beta = 0$ then the sum is just n , $n^{\Delta_0} = n$. If $\beta = 1$ then the sum is a triangle number, $n^{\Delta_1} = n^\Delta$. In general

$$\begin{aligned} n^{\Delta_\beta} &= \sum_{j=1}^{n} [1 + (j-1)\beta] \\ &= \sum_{j=1}^{n} [(1-\beta) + \beta j] \\ &= n(1-\beta) + \beta n^\Delta \end{aligned}$$

(14) $$\begin{aligned} n^{\Delta_\beta} &= n(1-\beta) + \beta n^\Delta \\ &= \frac{n(\beta n - \beta + 2)}{2} \end{aligned}$$

Letting $\beta = 2$ gives $n^{\Delta_\beta} = n^2$.

Before leaving this a few brief comments on pyramid numbers. Pyramids can have various polynomial bases. A number formed from a pyramid with equilateral sides and an equilateral base can be the sum of triangle numbers. For a square base the number can be formed from the sum of squares. Notation can be made and this defined as

(15)
$$\boxed{\begin{aligned} N^{\Delta^{\Delta}} &= 1 + 2^{\Delta} + 3^{\Delta} + \cdots + N^{\Delta} \\ N^{\Delta^{2}} &= 1 + 2^{2} + 3^{2} + \cdots + N^{2} \end{aligned}}$$

Where the Δ and the 2 are for triangle and square bases of the pyramid. It can be shown that

$$N^{\Delta^{2}} = 2N^{\Delta^{\Delta}} + N^{\Delta}.$$

Triangle numbers have been extended to real numbers, and can further be extended to complex numbers. Unfortunately, when dealing with complex numbers a simple visual picture cannot be made. Abstractions have to be made, which are not what the Math Drooler likes to play with. Below, the relation for a triangle number will be used without reference to visualization.

Any natural number, positive or negative, multiplied by itself yields a positive number. This can be written $x^2 = n \geq 0$. What if $n = -1$? The number x will not be a natural number but could surely exist. It is given the letter i and is defined by $i = \sqrt{-1}$. As when taking the square of a natural number, taking the triangle of a natural number always yields a positive number. What if $k^{\Delta} = -1$? Then

$$k = (-1)^{1/\Delta}$$

$$k^{\Delta} = -1 = \frac{k(k+1)}{2}$$

$$k^2 + k + 2 = 0$$

(15)
$$k = \frac{-1 \pm i\sqrt{7}}{2}$$

Using k instead of i, complex numbers can be written

$$n = a + kb = (a, b)_{\Delta}$$

The relation between triangular complex numbers and normal complex numbers is

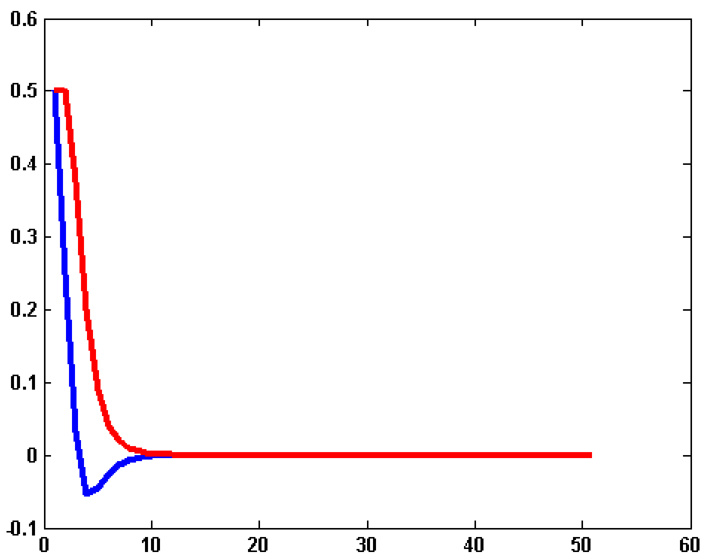
$$(a, b)_\Delta = \left(a - \frac{b}{2}, \pm\frac{\sqrt{7}}{2}b\right)$$

Now, consider a complex number $n = x + iy$. Using the standard equation for a triangular number

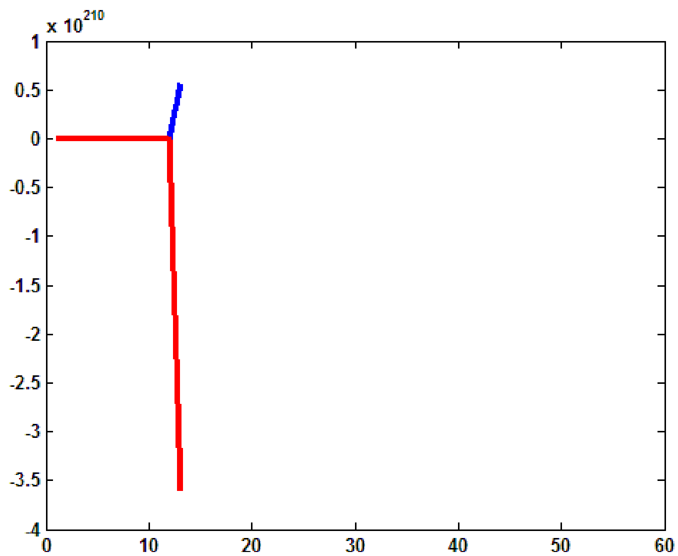
$$\begin{aligned} n^\Delta &= \frac{(x+iy)(x+iy+1)}{2} \\ &= \frac{x^2 - y^2 + x}{2} + i\frac{(2xy + y)}{2} \\ &= \frac{(x^2 + x) - (y^2 + y) + y}{2} + i\left(xy + \frac{y}{2}\right) \\ &= \left(x^\Delta - y^\Delta + \frac{y}{2}\right) = i\left(xy + \frac{y}{2}\right) \end{aligned}$$

(16)
$$\boxed{n^\Delta = \left(x^\Delta - y^\Delta + \frac{y}{2}\right) = i\left(xy + \frac{y}{2}\right)}$$

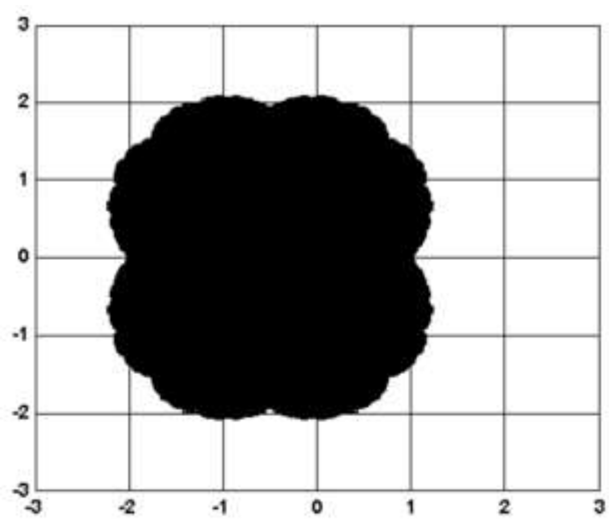
The fun comes when we pull out our math software and make some pictures. Take a complex number z_1 and do a triangle operation on it obtaining a complex number z_2. Do a triangle operation on that number obtaining z_3 and so forth. To see where this leads consider a few different starting points and do 20 iterations. Start with $z_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$. Using the math software the following plot is generated. The blue line is $real(z)$ and the red line is $imag(z)$.



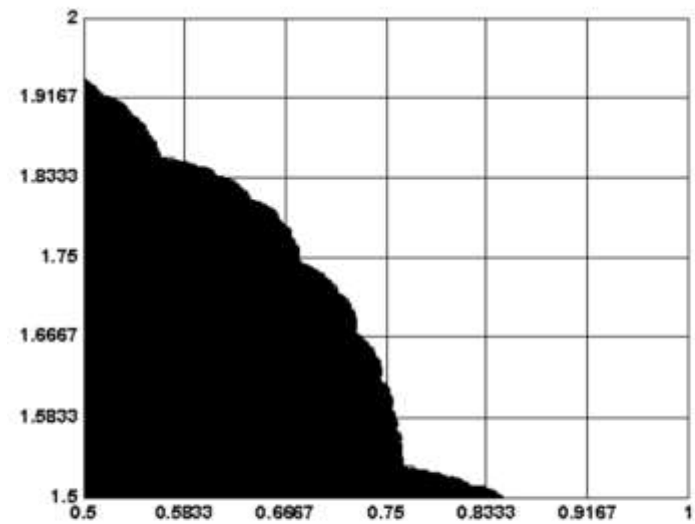
We see that both the real and imaginary parts of z tend to zero. Not real interesting. Using the math software again start with $z = (1.2, 1.5)$. We get the plot below.



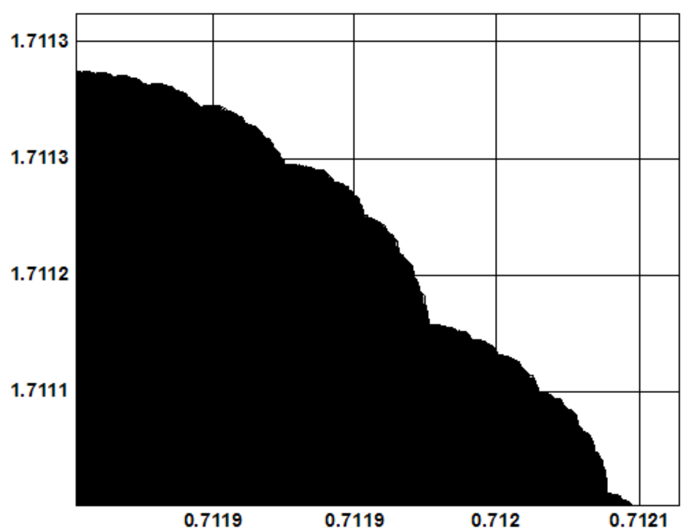
Now the real part of z went to infinity and the imaginary part went to negative infinity. Again this is not too interesting. Out of curiosity make a 2D plot of those places where the iteration on z goes to 0 and those places where it goes to plus or minus infinity. Take an area from $-3\ to\ 3$ for both the real and imaginary part of z. Just looking at the imaginary part of z, the math software gives the following plot where the black area is where the iteration goes to zero and the white area where it goes to plus or minus infinity.



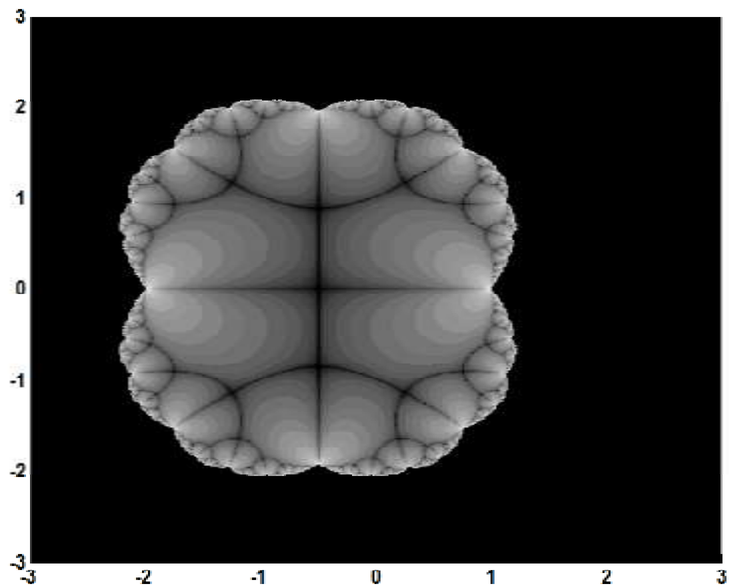
It is not clear what was expected, but it was not this. Something better defined would be nice. The shape of this fluff and its position are curious. It appears to be centered about $z = (0, -\frac{1}{2})$. Where did the $-\frac{1}{2}$ come from? The fluff is symmetric about the lines $x = 0$ and $y = -\frac{1}{2}$. It looks as though one could come up with a polar function about the point $(0, -\frac{1}{2})$ to describe the circumference of the fluff. To get a better look at the boundary take the interval for x to be from 0.5 to 1 and for y from 1.5 to 2. The plot below is generated.



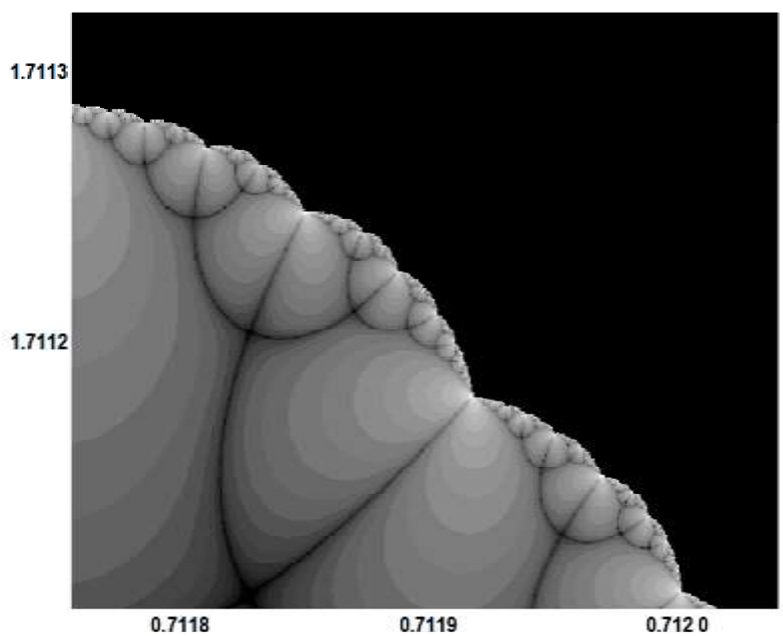
It still looks bumpy. Expanding an area several times, getting ever greater resolution, gives the following plot.



No matter how much an area is expanded, it seems to look the same. Go back to the first fluff and do something a little different. Not only show were the iteration goes to zero or infinity, but also show how many iterations it takes to get there. Remember only the imaginary part is being plotted.



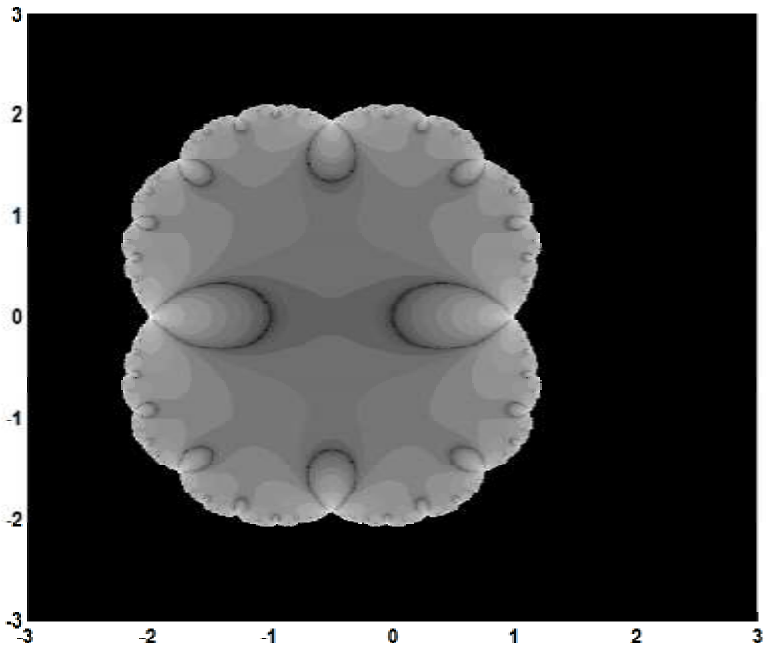
This looks more interesting. Looking at an area above that is greatly expanded gives the following.



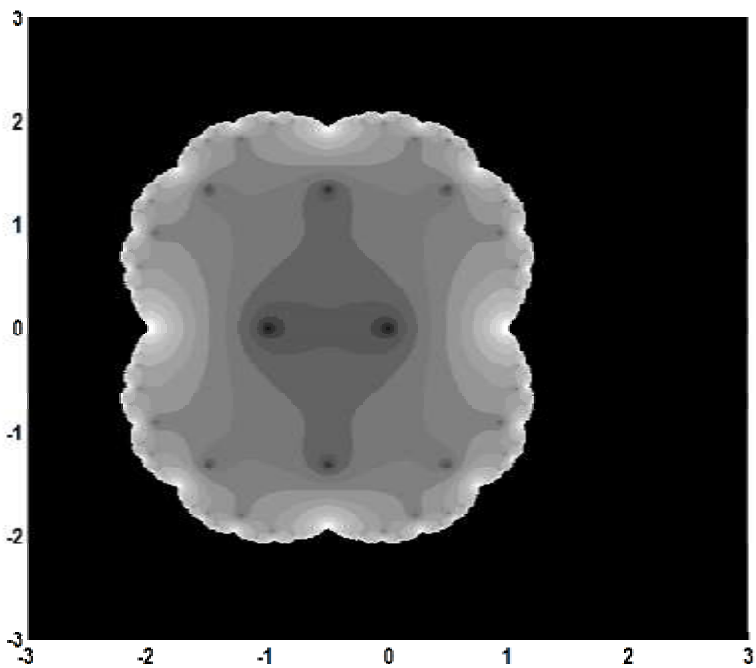
No matter how great the resolution or expansion at the boundary, the area looks the same. This is what is called a fractal.

The plots were of the imaginary part of z. Below are plots for $real(z)$ and $abs(z)$, respectively.

real(z)



abs(z)



While the plots look pretty, the main interest is in the boundary, the difference between where the iteration goes to zero and where it goes to plus or minus infinity. The plots give an indication of the rate at which the iteration goes to these limits, mainly showing the approach to zero.

Hopefully, a little more is known about triangle numbers from a simple concept with a few interesting properties that were maybe unexpected. The mathematical landscape is very diverse with the possibility of something of interest around any corner.

Math Drooler 3 Sep 2008
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