Weak Equal Sign Revisited

While browsing the book store I came across Roger Penrose's book "The Road to Reality – A Complete Guide to The Laws of the Universe". A thousand pages of Penrose for $25 I could not pass up. It will be a long read. As I read it hopefully ideas will pop into me head that I can pass along, assuming anyone reads this site. As for math and physics for the masses, I group Roger Penrose with Richard Feynman as to clarity and interest. I would place no one else in that group of author's I have read. That said, my first comment will be a criticism, hopefully enlightening some.

I spoke of this before but will reiterate with a little more emphasis. Penrose presents the equation

$$1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2} \tag{1}$$

He shows how one encounters problems when $x > 1$. The equation is composed of numbers $(1, 2, 4 \ldots)$, a variable (x), some operators $(+, -)$, an ellipses $(\ldots)$ and an equal sign $(=)$. The problem is that the preceding sentence is not correct. There is no ellipsis and no equal sign in the equation. There are the numbers, a variable, operators and "$\cdots =$", which is what a mathdrooler calls the weak equal sign. This is one symbol, not two. It is a magician's symbol, not that of a mathematician. The sleight of hand is that the eyes are taken off the ellipsis and the equal sign appears to stand alone, being the strong equal sign " $=$ ". The actual and correct equation is

$$1 + x^2 + x^4 + \cdots + x^{k-2} + \frac{x^k}{1-x^2} = \frac{1}{1-x^2}. \tag{2}$$

The ellipsis is separated from the equal sign, which is now the strong equal sign. There is no problem with $x > 1$. The only problem is when $x = 1$. Multiplying through by $1 - x^2$ helps alleviate in our minds that problem. This equation is correct but perhaps better written where there is no ellipsis.

$$(3) \qquad \sum_{i=0}^{k-1} x^{2i} + \frac{x^{2k}}{1 - x^2} = \frac{1}{1 - x^2}.$$

Ellipse is a convenient tool to help visualize a relationship as long as it does not make up the weak equal sign.

I showed in "dynamic series" how slight-of-hand artists use the weak equal sigh to make game of $\ln(2)$. Years ago I played with something called Reimann's Hypothesis. It also involves the weak equal sign. What I wrote is complex for a mathdrooler and I have to go over it slowly to understand it. One of the big questions in math is whether Reimann's Hypothesis is true or not. I think the conclusion from the perspective of the weak equal sign is that the question of the truth or falsehood of the hypothesis is not a legitimate inquiry in rigorous math. What was done may be presented in the future.

Before leaving, a little more clarity needs to be given. Rewrite Equation (2) as follows.

$$(4) \qquad Q_k - R_k = \frac{1}{1 + x^2} \; where$$

$$Q_k = 1 + x^2 + x^4 + \cdots + x^{2k} \quad and$$

$$R_k = \frac{x^{2k}}{x^2 - 1}.$$

As k tramps off to infinity for $x > 1$ one has

$$\infty_Q(x) - \infty_R(x) = \frac{1}{1 + x^2}.$$

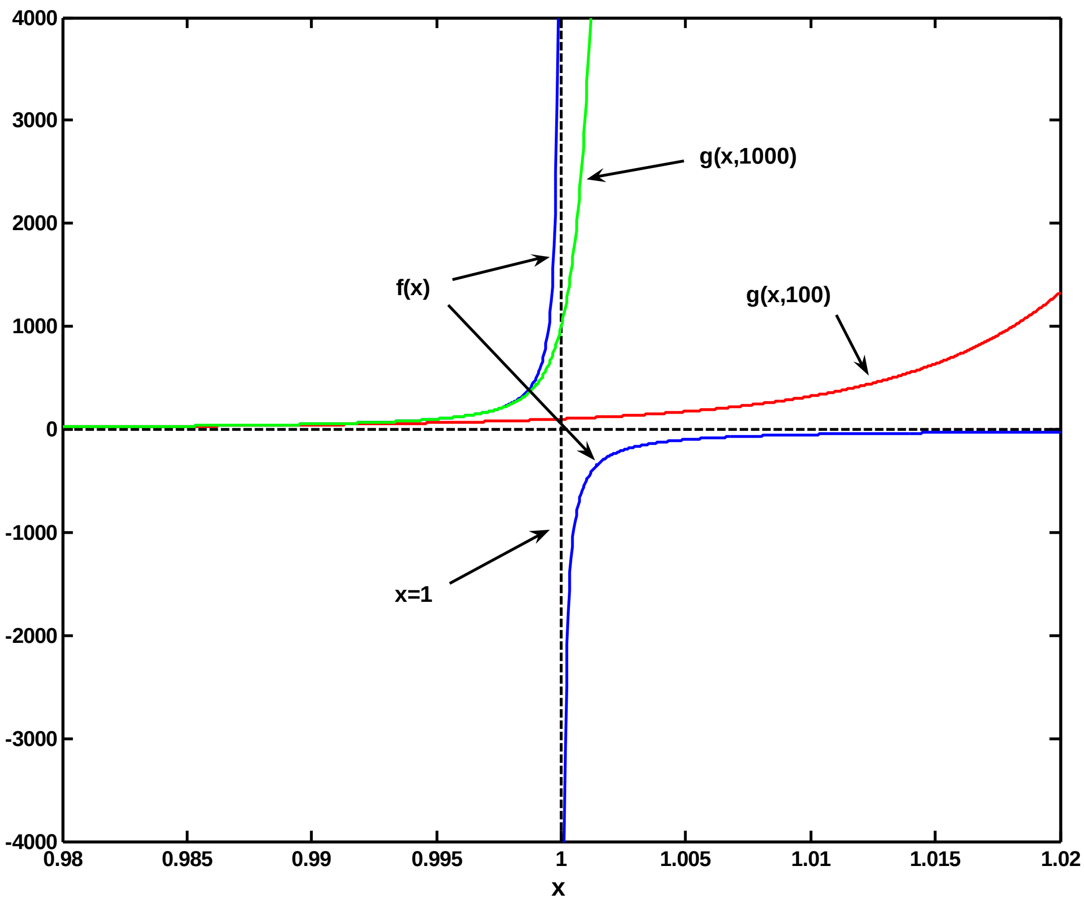
The notation may look strange. Both $Q_k \; and \; R_k$ are marching off into infinity. We are not privileged to see the infinite. But we can see the trek to it. That trek is not always, is rarely, at the same pace or the same place. For $x > 1$, $\infty_Q(x)$ and $\infty_R(x)$ are moving at a different pace and different place, being separated by $\frac{1}{1+x^2}$. Both $\infty_Q(x)$ and $\infty_R(x)$ are alive and dynamic.

Plotted in the figure below are

$$f(x) = \frac{1}{1-x^2} \qquad \textit{blue curve}$$

$$g(x,100) = \sum_{i=0}^{100} x^{2i} \qquad \textit{red curve}$$

$$g(x,1000) = \sum_{i=0}^{1000} x^{2i} \qquad \textit{green curve}.$$



No matter how large k, $g(x,k)$ crossed the line $x = 1$ and continues on its way. The function $f(x)$ never crosses $x = 1$. f and g are not the same function. In the plot above they look quite different.

Penrose goes on by expanding the investigation to complex numbers. However, the hour is late and it gets dark early this time of year. So looking at the weak equal sign in the complex plane will have to be a hike in the lower hills of the math world on another day.